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Non-critical holography and four-dimensional CFT's with fundamentals

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Abstract: We find non-critical string backgrounds in five and eight dimensions, holographically related to four-dimensional conformal field theories with $\mathcal{N} = 0$ and $\mathcal{N} = 1$ supersymmetries. In the five-dimensional case we find an AdS_5 background metric for a string model related to non-supersymmetric, conformal QCD with large number of colors and flavors and discuss the conjectured existence of a conformal window from the point of view of our solution. In the eight-dimensional string theory, we build a family of solutions of the form $AdS_5 \times \tilde{S}^3$ with $\tilde{S}^3$ a squashed three-sphere. For a special value of the ratio N_f/N_c, the background can be interpreted as the supersymmetric near-horizon limit of a system of color and flavor branes on $\mathbb{R}^{1,3}$ times a known four-dimensional generalization of the cigar. The $\mathcal{N} = 1$ dual theory with fundamental matter should have an IR fixed point only for a fixed ratio N_f/N_c. General features of the string/gauge theory correspondence for theories with fundamental flavors are also addressed.

Keywords: AdS-CFT and dS-CFT Correspondence, Supergravity Models.

Contents

1. Introduction

The string/gauge theory correspondence [1−3] postulates the identity of string theory on a non-trivial background and a related field theory living on the boundary of the space. In many important examples, most notably $\mathcal{N} = 4$ super Yang-Mills and quiver superconformal gauge theories, the string theory dual lives in ten dimensions and there exists a range of parameters for which the background has small curvature compared to the string length, $\sqrt{\alpha'}$. Critical string theory is then reliably approximated by supergravity, faithfully reproducing the dual gauge theory at large 't Hooft coupling.

There are, however, important obstructions that forbid the construction of a weakly curved ten-dimensional string theory dual for every gauge theory. For many phenomenologically interesting field theories, critical string theory has too many degrees of freedom to be able to reproduce the dynamics of the would-be dual field theory. In particular, the Kaluza-Klein states on some or all of the internal space directions couple to modes that do not belong to the desired field theory. In order to decouple these unwanted modes, one has to take a limit in which the string theory geometry becomes highly curved and stringy dynamics is no longer negligible. This is the case for example for pure (super) Yang-Mills theories [4−6], and it is expected for (large N_c) QCD.

This obstruction is not limited to confining theories. In fact, there are also conformal theories for which a weakly coupled supergravity dual does not exist, as for example all conformal theories whose central charges a and c differ at leading order [7]. Among these, there are some of the most interesting conformal theories, such as non-supersymmetric conformal QCD, the IR fixed points of $\mathcal{N} = 1$ theories containing fundamental matter, and $\mathcal{N} = 2$ superconformal QCD.

For all these theories it is expected that the string dual is a non-critical string theory. Of course it is in general not known how to handle string theory off criticality. Moreover, in principle, the gravity approximation to non-critical string theory is not reliable because of the unavoidable large curvature of its background. Nevertheless, the gravity approximation has proven, from the first days of the string/gauge theory correspondence, to give remarkable qualitative insights, in particular for highly symmetric examples [8]. More specifically, string theory on AdS vacua dual to conformal theories is believed to be subject to stringy corrections which do not spoil the solution but only affect its parameters [9]. This is the reason why it is very fruitful to study examples of non-critical strings/conformal gauge theory pairs: the quest for a string theory dual of QCD or pure Yang-Mills necessarily requires the comprehension of all mechanisms on which the non-critical string/gauge correspondence is based, which can be better understood starting from the study of simpler AdS/CFT pairs.

Some examples of non-critical AdS/CFT pairs for theories with flavors have recently been studied [10, 11][1]. In particular [10] considers $\mathcal{N} = 1$ QCD in Seiberg's conformal window. This example is particularly interesting because the IR field theory is conformal only when the ratio of the number of fundamental flavors to the number of colors is in the range $3/2 < N_f/N_c < 3$. Therefore fundamental flavors essentially contribute[2] to the string background being AdS. The setting considered in [10] is six-dimensional string theory on $\mathbb{R}^{1,3} \times \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$; the color Yang-Mills theory lives on D3-branes placed at the tip of the cigar (the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset manifold), while flavors are introduced by means of space-time filling uncharged D5-branes. The effect of the flavor branes is encoded by the addition of a Dirac action to the closed string effective action. This setting has been further studied in [16, 17].

[1]For other recent non-critical solutions see also [12, 13].

[2]Understanding flavors in string duals beyond the quenched approximation introduced in [14] is still an open problem in ten dimensions, see [15] for works in this direction.

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In this paper we further explore some new directions in the context of the non-critical gauge/gravity duality. Backgrounds with a non-trivial dilaton play a crucial role in this setting, because they provide good vacua for lower-dimensional string theories. D-branes placed on these backgrounds give rise to gauge theories in a way similar to what happens in the context of critical theories. We address two different examples of non-critical holographic pairs, involving in one case a five-dimensional string theory, and in the other an eight-dimensional one. As usual for non-critical solutions, the backgrounds we build have curvature of order one in units of α'. We focus on duals of conformal field theories, so that, as explained above, the Anti de Sitter structure of our solutions ensures that the two-derivative approximation captures the qualitative features of the full string models.

The five-dimensional backgrounds we find are a class of AdS_5 solutions[3]. They can be thought of as originating from the back-reaction of color D3 and flavor, space-time-filling, uncharged D4-branes on a linear dilaton background. The solutions have a natural interpretation as the duals of non-supersymmetric QCD with a large number of colors and massless flavors, in the conjectured conformal window [19]. We also analyze the spectrum of mesons by studying the fluctuations of the world-volume tachyon and gauge field of the flavor D4-branes. Since the conformal dimension associated to these operators depends on $\frac{N_f}{N_c}$, we argue that the limits of the conformal window should appear when some of these modes become tachyonic or reach the unitarity bound. However, a quantitative check is not possible due to $\mathcal{O}(1)$ string corrections.

For the eight-dimensional case, we consider a vacuum which is the direct product of four-dimensional flat Minkowski and a four-dimensional Kählerian space constructed in [20], denoted by KKL_4 in the following. This space has U(2) isometry and it originates from an $\mathcal{N}=2$ world-sheet SCFT. The number of space-time supercharges for the string model on the above vacuum is eight. This vacuum is a natural higher dimensional generalization of the two-dimensional cigar.

We look for an eight-dimensional background which can be interpreted as being sourced by N_c color branes with D3-charge placed at the tip of KKL_4 and N_f suitably distributed, uncharged flavor D5-branes. The back-reacted geometry turns out to be of the form $AdS_5 \times \tilde{S}^3$, where the internal $\tilde{S}^3$ is a squashed three-sphere, with an $\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry. Although we build an infinite set of such solutions (depending on N_f and N_c), we find that there is a special value of $\frac{N_f}{N_c}$, and present arguments that suggest that this is the only supersymmetric solution of the set. The exact form of the $\mathcal{N}=1$ SCFT dual to the above particular solution is still to be found.

Notice that non-critical string theory in eight dimensions is particularly interesting since, as argued in section 5, we expect the duals of $\mathcal{N}=2$ SYM and $\mathcal{N}=2$ SQCD to be constructed by placing branes on $\mathbb{R}^{1,5}\times cigar$. However, the fact that the full $\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry of such gauge theories may not be reflected in the geometry hinders the computation of the back-reacted solution. We leave this problem for future work.

On the same line, another eight-dimensional vacuum that would deserve attention is $\mathbb{R}^{1,1}\times\mathrm{KKL}_6$. One can think of placing color and flavor branes in it in order to build the dual

[3]For the connection of higher spin theories in AdS_5 with gauged vector models, see [18].

of an $\mathcal{N} = (1, 1)$ gauge theory with fundamentals in 1+1 dimensions. The back-reacted solution should be of the form $AdS_3 \times X_5$ where $X_5 \sim \mathrm{SU}(3) \times \mathrm{U}(1)/(\mathrm{SU}(2) \times \mathrm{U}(1))$ is a symmetric space with U(3) isometries. Its study, however, is beyond the scope of the present paper.

The plan of the paper is as follows. In section 2 we deal with the five-dimensional setup. In section 3 we discuss the $AdS_5 \times \tilde{S}^3$ solutions. We also study (part of) the spectrum of fluctuations with respect to small perturbations of the metric and dilaton and we discuss their stability. In section 4 we give some evidence signaling the special $\frac{N_f}{N_c}$ ratio. In particular, we employ a first order formalism that was first developed in the study of domain wall solutions in five-dimensional supergravity [21, 22] and show that only for a particular value of this ratio it is possible to write an explicit, simple superpotential. We also build a flow from this $AdS_5 \times \tilde{S}^3$ solution to the string vacuum described above. Section 5 is devoted to discussing the open problems in the field theory interpretation of the above particular solution.

In section 6 we consider some general problems of the non-critical AdS/CFT correspondence, of which this paper gives some examples. In particular, by examining field theory central charges, we stress the fact that generic superconformal theories with fundamental matter and N_f of the same order as N_c cannot have a weakly coupled supergravity dual. The same considerations should extend to non-supersymmetric theories as well.

We conclude with a summary and outlook in section 7. In four appendices we collect some results and checks on the solutions built in the main text. In particular, in appendix D we discuss the semi-classical embedding in the KKL_4 vacuum of the branes we use to build the $AdS_5 \times \tilde{S}^3$ solutions.

2. The five-dimensional solutions and conformal QCD

In this section we will present the simplest conformal solution incorporating the back-reaction of flavor branes. It is a simple generalization of the AdS_5 solution found by Polyakov in [8], but the inclusion of flavors provides some interesting new physics. The dual field theory is expected to be simply QCD with enough flavors in the fundamental (and anti-fundamental) to be at a conformal point [19].

The string-frame action in 5d is [23, 9] (with $\alpha' = 1$):

$$S = \frac{1}{2\kappa_{(5)}^2} \int d^5x \sqrt{-g_{(5)}} \left(e^{-2\phi} \left(R + 4(\partial_\mu \phi)^2 + \mathfrak{c} \right) - F_{(5)}^2 - 2Q_f e^{-\phi} \right), \tag{2.1}$$

where $\kappa_{(5)}$ is the five-dimensional Newton constant, $\mathfrak{c} = 10 - D = 5$ is the non-criticality constant needed to cancel the central charge and preserve the world-sheet conformal symmetry, $F_{(5)}^2 \equiv \frac{1}{5!} F_{\mu\nu\rho\sigma\lambda} F^{\mu\nu\rho\sigma\lambda} \sim Q_c^2$ with Q_c being proportional to the rank N_c of the gauge group, and Q_f is proportional to the number of fundamental flavors N_f. We are thinking about three-branes and space-time filling, uncharged four-branes giving the Q_f-dependent term. The latter are the analogue of the uncharged five-branes introduced in [10]. In that case, they have been argued in [16] to be ordinary branes double-sheeted on the cigar. Here, on the other hand, the four-branes should be most probably viewed as brane-antibrane pairs in order to be uncharged.

We will search for a conformal solution having constant dilaton and string frame metric:

$$ds^2 = e^{2kr} dx_{1,3}^2 + dr^2 \,. \tag{2.2}$$

The equations of motion for the constant dilaton ansatz are:

$$R = -\mathfrak{c} + Q_f e^{\phi} \,, \tag{2.3}$$

$$R_{\mu\nu} = \frac{g_{\mu\nu}}{2}(R - 2Q_f e^{\phi} + \mathfrak{c}) + \\ + e^{2\phi}\frac{1}{4!}(F_{\mu\mu_2\ldots\mu_5}F_{\nu}{}^{\mu_2\ldots\mu_5} - \frac{g_{\mu\nu}}{10}F_{\mu_1\ldots\mu_5}F^{\mu_1\ldots\mu_5}) \,, \tag{2.4}$$

$$\partial_\mu\left(\sqrt{-g_{(5)}}F^{\mu\mu_2\ldots\mu_5}\right) = 0 \,. \tag{2.5}$$

These equations have a simple AdS_5 solution with five-form $F_{(5)} = Q_c Vol(AdS_5)$. In fact, the dilaton and Ricci tensor read:

$$e^{\phi} = \frac{\sqrt{49Q_f^2 + 40\mathfrak{c}Q_c^2} - 7Q_f}{10Q_c^2} \,, \qquad R_{\mu\nu} = -\frac{g_{\mu\nu}}{2}\left(Q_f e^{\phi} + Q_c^2 e^{2\phi}\right) \,, \tag{2.6}$$

that is, the space is of constant curvature with radius $R_{AdS}^2 = 1/k^2$:

$$R_{AdS}^2 = \frac{200Q_c^2}{10\mathfrak{c}Q_c^2 + 7Q_f^2 - Q_f\sqrt{40\mathfrak{c}Q_c^2 + 49Q_f^2}} \,. \tag{2.7}$$

The R_{AdS}^2 varies monotonically between $20/\mathfrak{c}$ and $28/\mathfrak{c}$, when Q_f/Q_c varies from zero to infinity. For large Q_c and Q_f the dilaton is always small, while the 't Hooft coupling $\lambda \sim e^{\phi}Q_c$ is of order one. For $Q_f = 0$ we end up with the Polyakov solution [8].

The interesting feature of this solution is that, unlike the solution in [10], the radius is not independent of Q_c and Q_f. This fact implies that the mass of the open string tachyon of the system of D4-anti D4 branes depends on Q_c and Q_f too, so that it can give constraints on their allowed values, as we will discuss in section 2.1.

The spectrum of five-dimensional string theory includes an axion and a one-form potential [23]. In fact, a constant axion can always be included in the solution above. It is as usual dual to the θ_{YM} term. The baryonic $\mathrm{U}(1)_B$ should be given by the one-form potential; the latter couples to a D0-brane which should be dual to the baryon vertex, but the details of this identification require a deeper analysis. The axial $\mathrm{U}(1)_A$ can be identified with the symmetry of the complex tachyon on the D4 anti-D4 pairs. The non-abelian $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ flavor symmetry is as usual the gauge symmetry on the D4's.

Inclusion of the closed string tachyon

Generically, the spectrum of non-critical closed strings contains (at least) one tachyonic scalar T. Since T couples to the other fields, it should be included in the gravity action. The action is a simple generalization of (2.1):

$$S = \frac{1}{2\kappa_{(5)}^2}\int d^5x\sqrt{-g_{(5)}}\left(e^{-2\phi}(R + 4(\partial_\mu\phi)^2 + \mathfrak{c}(T) - \frac{1}{2}(\partial_\mu T)^2) - f(T)F_{(5)}^2 - 2Q_f g(T)e^{-\phi}\right) \tag{2.8}$$

where $\mathfrak{c}(T) = 5 + \frac{3}{8}T^2 + \cdots$, $f(T) = 1 + O(T)$ and $g(T) = 1 + O(T)$ are known only up to the first few orders in their expansion around the zero-tachyon value. There are linear terms in the tachyon in $f(T)$ and $g(T)$ for type 0 theories but not for type-II theories. As usual, we have to assume that the tachyon expectation value is small to keep it under control.

The equations of motion (2.3)-(2.5) have the obvious generalization given by the inclusion of the tachyon factors, but still have the AdS_5 solution, with constant tachyon $T = T_0$ given by its equation of motion (here the prime means derivative w.r.t. the tachyon field):

$$0 = \mathfrak{c}'(T_0)e^{-2\phi} - 2g'(T_0)Q_f e^{-\phi} - f'(T_0)F_{(5)}^2 \,. \tag{2.9}$$

Type-II theories admit the $T_0 = 0$ solution above. In type 0 theories this is not the case, but given a solution of (2.9), one has to simply replace in the solution (2.6), (2.7), $\mathfrak{c} = 5$ with $\mathfrak{c}(T_0)$, Q_c^2 with $Q_c^2 f(T_0)$ and Q_f with $Q_f g(T_0)$.

2.1 Fluctuations of the flavor branes

We will now briefly consider the quadratic fluctuations of the flavor branes, which will give us information on the spectrum of mesons in the dual field theory. The fluctuations will be limited to the gauge fields on the branes and to the open string tachyon of the brane-anti brane pairs. We will concentrate on the abelian case, that is, a single D4-anti D4 brane pair. The action for small open string tachyon τ is [24]:

$$S = -\int d^5\xi\, V(|\tau|)e^{-\phi}(\sqrt{-detX^1} + \sqrt{-detX^2})\,, \tag{2.10}$$

with:

$$V(|\tau|) = 1 - \frac{\pi}{2}|\tau|^2 + \cdots\,, \qquad X^{1,2}_{ab} = g^{1,2}_{ab} + 2\pi F^{1,2}_{ab} + \pi(\partial_a\tau(\partial_b\tau)^* + \partial_b\tau(\partial_a\tau)^*)\,. \tag{2.11}$$

Since the tachyon enters already quadratically, its fluctuations do not mix with the ones of the gauge fields. Moreover, the two gauge fields $A^{1,2}$ are also decoupled to this order and have of course the same fluctuations. Therefore, the fluctuating fields are just two free massless gauge fields (the A_r component does not give a regular mode dual to a meson) and one massive (with bare mass $m^2 = -1/2$) free complex scalar. The conformal dimensions of the dual operators are given by the standard AdS/CFT formula: $\Delta = 3$ for the massless gauge fields and

$$\Delta_\pm = 2 \pm \sqrt{4 - \frac{R^2_{AdS}}{2}} \tag{2.12}$$

for the complex tachyon. Note that the latter is no more tachyonic, due to the curvature of the background, if $R^2_{AdS} \leq 8$. The boundary value $R^2_{AdS} = 8$ corresponds to the dimension $\Delta = 2$. As we decrease R^2_{AdS}, we can have the two possible conformal dimensions $\Delta_\pm$ until we reach the value $R^2_{AdS} = 6$, where Δ_- hits the unitarity bound (and $\Delta_+ = 3$). For smaller values of the radius we have only $\Delta_+ > 3$.

The interpretation of this spectrum is straightforward. The gauge fields are dual to the vector mesons. The vectorial current is exactly conserved, so the dimension is protected by conformal symmetry and it is equal to the free one, namely $\Delta = 3$.

For what concerns the fluctuations of the complex tachyon, they are dual to the scalar mesons. The dimensions of these fields are not protected and differ from the free ones. Conformal QCD is expected to have a conformal window as its $\mathcal{N} = 1$ counterpart. The supposed conformal window for this theory starts at a ratio N_f/N_c which lies between $5/3 \sim 1.7$ and 4, and it ends surely before (or at) $N_f/N_c = 11/2 = 5.5$ (see for example [25]). In analogy with the $\mathcal{N} = 1$ theory, the dimensions of the scalar mesons are expected to violate the unitarity bound when we try to go outside this conformal window.

In fact, and unlike the solution dual to $\mathcal{N} = 1$ SQCD [10], we find that these dimensions depend on N_f and N_c through R^2_{AdS} (cfr. (2.7)). In this sense, this solution seems to give a confirmation of the expectations above. Unfortunately, the fact that the gravity solution gets corrections of order one to the radius of AdS (as usual for non-critical solutions) does not allow us to perform a meaningful numerical check of this picture. In particular, if we could quantitatively trust (2.7) we would get $4 \leq R^2_{AdS} \leq 28/5 = 5.6$ for all values of Q_f/Q_c, so that the only conformal dimension would be Δ_+ (2.12) and it is always greater than three, thus not giving any conformal window. The addition of the closed string tachyon, if we insist in keeping it small, cannot improve substantially the picture, since it only gives a small correction to the numbers in R^2_{AdS} (2.7). Clearly, these numbers would contradict the observation above that the conformal window should be bounded by values of N_f/N_c corresponding to $R^2_{AdS} = 6$ and $R^2_{AdS} = 8$. We nevertheless expect that in the complete (string corrected) solution the value of the AdS radius will be shifted, singling out the values of N_f/N_c corresponding to the conformal window. We can in fact expect various types of corrections to the numbers we get. The first comes from a non-small closed string tachyon value, shifting $\mathfrak{c}(T_0)$, $f(T_0)$, $g(T_0)$ and then R^2_{AdS} by a relevant amount. Moreover, the latter can well be changed by α' corrections. Finally, we can expect that the effective mass of the open tachyon will get additional N_c, N_f dependence from some higher order couplings we neglected, since this is the way it is expected to single out the conformal window in the $\mathcal{N} = 1$ case [10].

3. $AdS_5 \times \tilde{S}^3$: setup and solutions

In this section we look for the solution obtained by considering the back-reaction of color and flavor branes in an eight-dimensional supersymmetric non-critical background, which can also be regarded as the simplest generalization of the cigar.

3.1 The setup

In [20] a sequence of solutions, KKL_{2n}, to the $2n$ dimensional string equations was found. They are Kählerian spaces of complex dimension n and isometry $\mathrm{U}(n)$. They have a non-trivial dilaton, but the string coupling is bounded from above. The first space in the series,

KKL_2, coincides with the well-known $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset CFT background, i.e. the 2d cigar with U(1) isometry. By suitably placing D-branes on $\mathbb{R}^{1,3} \times \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$, one can study the duals of $\mathcal{N}=1$ SQCD [10, 16] and $\mathcal{N}=1$ SYM [17].

Here we consider the eight-dimensional vacuum $\mathbb{R}^{1,3}\times \mathrm{KKL}_4$:

$$ds_8^2 = dx_{1,3}^2 + ds^2_{\mathrm{KKL}_4}\,, \tag{3.1}$$

where:

$$\begin{aligned} ds^2_{\mathrm{KKL}_4} &= \frac{1}{4f_2(\zeta)}d\zeta^2 + \frac{\zeta}{4}(d\theta^2+\sin^2\theta\, d\varphi^2) + \frac{f_2(\zeta)}{4}(d\psi+\cos\theta\, d\varphi)^2\,, \\ \phi &= -\frac{\zeta}{K} + \mathrm{const}\,. \end{aligned} \tag{3.2}$$

The ranges for the angles are $\theta\in[0,\pi]$, $\varphi\in[0,2\pi)$, $\psi\in[0,4\pi)$ and the function $f_2(\zeta)$ reads:

$$f_2(\zeta) = \frac{K^2}{2}\frac{e^{-\frac{2\zeta}{K}}-1+\frac{2\zeta}{K}}{\zeta}\,. \tag{3.3}$$

For any value of ζ, the angular part of the KKL_4 metric describes a squashed three-sphere, also called Berger sphere. This homogeneous and anisotropic space, which we will denote by $\tilde{S}^3$, can be described as a coset manifold $\frac{\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{U}(1)}$ or as an S^1-fibration over S^2. It has the same topology as the round three-sphere but its isometry group is just $\mathrm{SU}(2)\times\mathrm{U}(1)$. A review of several features of its geometry can be found in [26].

In (3.3) we fixed an integration constant so that the metric is smooth at the tip $\zeta=0$ [27]. In fact, near that point, the metric reduces to flat $\mathbb{R}^4$. The constant K is related to the background central charge that must be canceled. As the central charge deficit of the KKL_4 space is:

$$\delta c = \frac{24}{K}\,, \tag{3.4}$$

if one takes the product $\mathbb{R}^{1,D-5}\times \mathrm{KKL}_4$, it is necessary to set:

$$K = \frac{16}{10-D}\,, \tag{3.5}$$

and therefore $K=8$ in the case of our interest.

The interpretation of the KKL_4 background as the target space metric of a 2d $\mathcal{N}=2$ CFT is discussed in [27]. In that paper it is also shown that the KKL_4 model with constant $K=4m$ emerges in 10d from a set of m NS5 branes wrapped on the two-cycle of a CY_2. This preserves eight space-time supercharges. Our vacuum should thus preserve the same amount of supercharges.

To introduce colors, we place D3 branes at the tip of (3.2). This should break supersymmetry by one half, leaving $\mathcal{N}=1$ supersymmetry in 4d (which, as usual, can be enhanced to $\mathcal{N}=1$ superconformal symmetry if the solution contains an AdS_5 factor). To introduce flavors, we add a stack of N_f D5 branes filling all the non-compact directions of (3.1) and wrapping a contractible one-cycle inside the $\tilde{S}^3$. Concretely, we take the flavor branes to be extended along $x_0,\ldots,x_3,\psi$ and τ, where we have introduced a new radial

							$\tilde{S}^3$	
	$x_{1,3}$				τ	ψ	θ	φ
D3	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
D5	$-$	$-$	$-$	$-$	$-$	$\bigcirc$	$\sim$	$\sim$
	Gauge theory							

Table 1: Brane configuration in the general setup. A line $-$ means that the brane spans a non-compact dimension, a point $\cdot$ that it is point-like in that direction, a circle $\bigcirc$ that it wraps a compact cycle and a tilde $\sim$ that, although branes are not extended in that direction, they are uniformly distributed there.

coordinate $\tau \in (-\infty, \infty)$ ($\tau \to -\infty$ corresponds to the tip of KKL$_4$ and is the point where the D3s are sitting). These D5-branes are extended up to $\tau \to \infty$, so the gauge fields coming from the D5-D5 open strings are decoupled from the theory living on the D3-branes. The D3-D5 open string degrees of freedom are quarks transforming in the fundamental representation of the gauge group. In order to have gapless D3-D5 strings and therefore massless quarks, we take the D5 to be extended also up to $\tau \to -\infty$. Moreover, in order to preserve the SU(2) isometry of (3.2), they are homogeneously smeared on the transverse S^2 parameterized by θ and φ. Notice that this is only possible in the limit $N_f \to \infty$. The brane configuration is summarized in table 1.

The action we consider is the (bosonic) two-derivative approximation to the low energy eight-dimensional non-critical IIA superstring theory[4]. Keeping just the $F_{(5)}$ out of all possible RR field strengths, we write the following string frame action:

$$S = \frac{1}{2\kappa_{(8)}^2} \int d^8x \sqrt{-g_{(8)}} \left(e^{-2\phi} \left(R + 4(\partial_\mu \phi)^2 + 2 \right) - F_{(5)}^2 \right) + S_{\text{flavor}} \,. \tag{3.6}$$

We look for eight-dimensional solutions with the same SU(2)$\times$U(1) isometry as KKL$_4$, which can be interpreted as the geometry produced by placing on (3.1) the configuration of branes described above. The natural ansatz for the back-reacted metric is (in string frame):

$$ds_8^2 = e^{2\lambda}\, dx_{1,3}^2 + d\tau^2 + e^{2\nu}(d\theta^2 + \sin^2\theta d\varphi^2) + e^{2f}(d\psi + \cos\theta\, d\varphi)^2 \,, \tag{3.7}$$

where, in principle, the functions λ, ν, f depend on τ.

The stack of N_c D3 branes that supports the gauge theory is extended in the x_i-directions and is a source for the five-form RR field strength. The following ansatz for $F_{(5)}$ guarantees that its equation of motion is satisfied:

$$F_{(5)} = Q_c e^{4\lambda - 2\nu - f}\, dx^0 \wedge \ldots \wedge dx^3 \wedge d\tau \,, \tag{3.8}$$

where the constant Q_c is N_c up to some normalization factor.

[4]We denote by IIA theory, in any dimension, the string theory whose RR sector is the tensor product of two spinors of opposite chirality. In eight dimensions, this gives field-strengths that are one-, three-, five- and seven-forms.

The term S_{flavor} in (3.6) is the action for the flavor branes. We will consider their back-reaction on the geometry in a fashion similar to [10]. Their contribution is given by the Dirac action:

$$S_{flavor} = -T_5 \sum_{i=1}^{N_f} \int_{\mathcal{M}_6} d^6\xi \, e^{-\phi} \sqrt{-\hat{g}_{(6)}} \,, \tag{3.9}$$

where $\mathcal{M}_6$ is the brane world-volume and $\hat{g}_{(6)}$ stands for the determinant of the pull-back of the metric on such world-volume.

Because of the smearing, we can substitute the sum in (3.9) by an integral over the space transverse to the D5 branes: defining a constant density along the transverse S^2, we have $\sum_{i=1}^{N_f} \to \frac{N_f}{4\pi} \int d\theta d\varphi \sin\theta$. We arrive at the expression[5]:

$$S_{flavor} = -\frac{T_5 N_f}{4\pi} \int d^8x \frac{\sqrt{-g_{(8)}}}{e^{2\nu}} e^{-\phi} \,. \tag{3.10}$$

By defining, for notational convenience:

$$Q_f = \frac{\kappa_{(8)}^2 T_5}{2\pi} N_f \,, \tag{3.11}$$

and inserting the ansatz (3.7), (3.8), (3.10) in the action (3.6) (with the usual change of sign for the $F_{(5)}^2$ term), we obtain the effective action which will be the starting point of the analysis in the following sections:

$$\begin{aligned} L_{\text{eff}} = e^{4\lambda+2\nu+f-2\phi} \big[& 12\dot\lambda^2 + 2\dot\nu^2 + 16\dot\lambda\dot\nu + 8\dot f\dot\lambda + 4\dot f\dot\nu - 4\dot f\dot\phi - 16\dot\lambda\dot\phi - 8\dot\nu\dot\phi + 4\dot\phi^2 + 2 - \\ & - \frac{1}{2} e^{2f-4\nu} + 2e^{-2\nu} - Q_c^2 e^{2\phi-4\nu-2f} - Q_f \, e^{\phi-2\nu} \big]. \end{aligned} \tag{3.12}$$

From now on, a dot stands for the derivative with respect to τ. The consistency of this effective action approach is checked in appendix A, where it is shown that the solutions found from this lagrangian satisfy the whole set of Einstein equations.

For completeness, we write here the expressions for the curvature scalar and determinant of the metric (3.7) used to derive (3.12):

$$\begin{aligned} R &= -\frac{1}{2} e^{2f-4\nu} + 2e^{-2\nu} - 2\dot f^2 - 20\dot\lambda^2 - 16\dot\lambda\dot\nu - 6\dot\nu^2 - 8\dot f\dot\lambda - 4\dot f\dot\nu - 2\ddot f - 8\ddot\lambda - 4\ddot\nu \,, \\ \sqrt{-g_{(8)}} &= \sin\theta \; e^{4\lambda+2\nu+f} \,. \end{aligned} \tag{3.13}$$

3.2 General $AdS_5 \times \tilde{S}^3$ solutions

Since we are interested in the string duals of conformal field theories, the most natural ansatz has an AdS form:

$$\lambda = k_0 \tau \,, \qquad \phi, \, \nu \,, f = \text{constants} \,, \tag{3.14}$$

[5]Notice that the difference from taking D7 flavor branes which would fill the whole eight-dimensional space-time is the $e^{2\nu}$ in the denominator. No solutions of the type we are searching for can be found in this case; we will further comment on this fact in appendix A.

where the constant k_0 is the inverse of the AdS radius:

$$R_{AdS} = \frac{1}{k_0}\,. \tag{3.15}$$

By inserting this ansatz in the equations of motion derived from (3.12), one finds a two-parameter (depending on the $\frac{Q_c}{Q_f}$ ratio and also on Q_f through the dilaton) family of solutions:

$$ds_8^2 = e^{2k_0\tau} dx_{1,3}^2 + d\tau^2 + e^{2\nu_0}(d\theta^2 + \sin^2\theta\, d\varphi^2) + e^{2f_0}(d\psi + \cos\theta\, d\varphi)^2\,, \tag{3.16}$$

with:

$$e^{\phi_0} = \frac{2}{3}\frac{A}{Q_f}\,, \qquad e^{2f_0} = \frac{1}{2}\left(1 - \frac{A}{3}\right)(A+1)\,, \qquad e^{2\nu_0} = \frac{1}{2}(A+1)\,, \qquad k_0^2 = \frac{A+3}{12(A+1)}\,, \tag{3.17}$$

and the parameter A a root of the following cubic equation:

$$A^3 + \left(\frac{16}{3}\frac{Q_c^2}{Q_f^2} - 1\right) A^2 - 5A - 3 = 0\,. \tag{3.18}$$

This equation has three real roots (one positive and two negative ones) for every value of $\frac{Q_c}{Q_f}$. Notice, however, that, in order to give a sensible solution (3.17), the only allowed values of A lie in the range $0 \le A \le 3$. Notably, the positive root of (3.18) is a monotonic function of the ratio $\frac{Q_c}{Q_f}$ and it is easy to check that $A \to 3$ when $\frac{Q_c}{Q_f} \to 0$ and that $A \to 0$ when $\frac{Q_c}{Q_f} \to \infty$. Therefore, every value of $\frac{Q_c}{Q_f}$ yields a single $AdS_5 \times \tilde{S}^3$ metric and the squashing of the Berger sphere is a monotonic function of this ratio. In the flavorless limit $A \to 0$, $f_0 = \nu_0$ and the three-sphere becomes round; this is precisely the $AdS_5 \times S^3$ solution found in [12].

As generally expected for non-critical strings, all these solutions have string-scale curvatures and, unlike the usual critical cases, there is no limit in which they become weakly curved. Therefore, one should take into account higher order string corrections to the action. Nevertheless, as argued in [9, 12, 10], one can expect that the high symmetry of the AdS configurations (3.16) protects the form of the solutions, and that string corrections will, at most, change the radii by some factor of order one.

On the other hand, the dilaton is parametrically small, so the solutions do not have string loop corrections, as usual.

3.3 A special solution

The solutions displayed in section 3.2 were constructed without making any reference to the supersymmetric vacuum (3.1), (3.2). Actually, there is a very special solution among this family, which can be shown to be the result of piling up branes on the $\mathbb{R}^{1,3} \times \mathrm{KKL}_4$ vacuum (see section 4). This solution corresponds to taking:

$$\frac{Q_f}{Q_c} = \sqrt{\frac{2}{3}}\,. \tag{3.19}$$

Plugging (3.19) in (3.18), we find $A = 1$, so the gravity solution is given by (3.16) and (3.8), where the dilaton and the coefficients in the metric are (3.17):

$$e^{\phi_0} = \frac{2}{3Q_f}\,, \qquad e^{2f_0} = \frac{2}{3}\,, \qquad e^{2\nu_0} = 1\,, \qquad k_0^2 = \frac{1}{6}\,. \tag{3.20}$$

As it will be shown in section 4.1, this special configuration is the only one among (3.16)-(3.18) for which one can write a simple, explicit superpotential such that the solution satisfies the corresponding system of first-order BPS equations. In section 4.2 we will also build a numerical solution interpolating between this special AdS solution (3.20) in the IR and the $\mathbb{R}^{1,3}\times$ KKL$_4$ vacuum (3.1) in the UV. These facts support the conjecture that this solution is supersymmetric. For all other solutions, on the contrary, similar arguments cannot be made.

As we show in appendix C using a general argument of [12], we expect all solutions (3.16)-(3.18) to arise from a superpotential, but its form to be in general too complicated to be obtained analytically. Being the solution with $Q_f/Q_c = \sqrt{2/3}$ the only one to admit a simple superpotential, we argue that in some sense it is "the most symmetric" one, its larger symmetry constraining more strxongly the system and therefore making the associated superpotential likely to turn out to be easier to write. Here "symmetric" of course does not refer to the isometry of the background since all other solutions have the same isometry. It could well relate to supersymmetries. A similar argument for the identification of supersymmetric vacua was also suggested in [28].

We conjecture, therefore, that the special solution (3.20) is dual to an $\mathcal{N} = 1$ superconformal theory. This theory should be such that at the IR fixed point one particular ratio N_f/N_c is singled out among all other possibilities, as signaled by (3.19).

On general grounds, we expect that each of the solutions in (3.17) is dual to a distinct conformal gauge theory with SU(2) $\times$ U(1) global symmetry. We may however conclude, following [12], that either all other solutions are not stable, or they correspond to a family of unknown, possibly non-supersymmetric, conformal field theories with the aforementioned global symmetry. Notice that the existence of a superpotential, by itself, does not guarantee that a solution is supersymmetric [22].

In view of the field theory interpretation of the special solution (3.19), it is of evident interest to know what the relation (3.19) implies in terms of the actual number of D3 and D5 branes. Let us assume the usual quantization condition for the D3 brane charge:

$$\frac{1}{2\kappa_{(8)}^2}\int_{\tilde{S}^3} {}^*F_{(5)} = N_c T_3 = \frac{1}{2\kappa_{(8)}^2} 16\pi^2 Q_c\,. \tag{3.21}$$

For the first equality, we have assumed that, as in critical string theory, the charge and tension of the brane are equal, while the second equality is obtained by direct computation with the solution. From (3.11), (3.19) and (3.21), we then obtain:

$$\frac{N_f}{N_c} = \pi\sqrt{\frac{2}{3}} \simeq 2.56\,. \tag{3.22}$$

Notice that to derive this expression the ratio of tensions $\frac{T_3}{T_5} = (2\pi)^2$ has been used, but the precise expressions for T_3, T_5, $\kappa_{(8)}$ were not necessary.

Clearly, the above irrational value for $\frac{N_f}{N_c}$ cannot match the actual field theory value, which by definition has to be rational. One obvious possibility is that stringy corrections here are of order one. For instance, terms with higher powers of $F_{(5)}$ in the lagrangian would affect the relation between Q_c and N_c, and corrections to the DBI action would alter the value of $\frac{Q_f}{N_f}$. Furthermore, all sorts of corrections, like higher curvature terms, may in part modify the superpotential calculation of section 4, possibly leading to a value of $\frac{Q_f}{Q_c}$ different from (3.19). Finally, another possibility is that the normalization of the brane tensions we are using is incorrect in this type of background.

3.4 Spectrum of perturbations around $AdS_5\times\ \tilde{S}^3$ and stability

We consider now the spectrum of perturbations around the solutions of the form $AdS_5\times\tilde{S}^3$ we found in section 3.2. This is interesting mainly for two reasons: first of all, it will test the stability of these backgrounds against the excitation of some metric and dilaton modes[6]. Secondly, it will be essential for the construction of a flow connecting the special AdS solution of section 3.3 to the $\mathbb{R}^{1,3}\times$ KKL$_4$ vacuum, as we will see in section 4.

Let us start from the metric (3.7) and effective action (3.12) in the string frame. It is useful to define a new radial coordinate u and new fields α, β, γ:

$$\begin{aligned} d\tau &= e^{\frac{2}{3}\phi-\frac{2}{3}\nu-\frac{1}{3}f}\, du\,, \\ \alpha &= -\frac{2}{3}\phi+\frac{2}{3}\nu+\lambda+\frac{1}{3}f\,, \\ \beta &= -\frac{1}{3}\phi+\frac{2}{3}\nu+\frac{1}{3}f\,, \\ \gamma &= \frac{1}{3}\nu-\frac{1}{3}f\,, \end{aligned} \tag{3.23}$$

in terms of which the metric reads:

$$ds^2 = e^{\frac{2}{3}\phi-2\beta}(du^2+e^{2\alpha}dx_{1,3}^2)+e^{\frac{2}{3}\phi+2\beta+2\gamma}(d\theta^2+\sin^2\theta\, d\varphi^2)+e^{\frac{2}{3}\phi+2\beta-4\gamma}(d\psi+\cos\theta\, d\varphi)^2\,. \tag{3.24}$$

The field $\gamma(u)$ determines the squashing of the three-sphere: when $\gamma=0$ the three-sphere is perfectly round. The effective action becomes:

$$S_{\text{eff}} = 4\int du\, e^{4\alpha}\left(3\alpha'^2-\frac{1}{6}\phi'^2-\frac{3}{2}\beta'^2-\frac{3}{2}\gamma'^2-V(\phi,\beta,\gamma)\right)\,, \tag{3.25}$$

where $'$ denotes derivation with respect to u, and the potential V does not depend on $\alpha(u)$:

$$V(\phi,\beta,\gamma) = -\frac{1}{2}e^{\frac{2}{3}\phi-2\beta}+\frac{1}{8}e^{-4\beta-8\gamma}-\frac{1}{2}e^{-4\beta-2\gamma}+\frac{1}{4}Q_c^2e^{\frac{2}{3}\phi-8\beta}+\frac{1}{4}Q_fe^{\phi-4\beta-2\gamma}\,. \tag{3.26}$$

[6] The full action would also include the closed string tachyon. We do not discuss this mode since we lack a satisfactory description of it. Analogously to the six-dimensional case studied in [10], we expect that the closed tachyon will not cause any instability on our eight-dimensional backgrounds and that the analysis of this section is, at least, qualitatively correct.

It is straightforward to write the equations of motion for these new fields in terms of the radial coordinate u:

$$
\begin{aligned}
3\alpha'' + 6\alpha'^{\,2} + \frac{1}{3}\phi'^{\,2} + 3\beta'^{\,2} + 3\gamma'^{\,2} + 2V &= 0\,,\\
\phi'' + 4\alpha'\phi' &= 3\frac{\partial V}{\partial \phi}\,,\\
\beta'' + 4\alpha'\beta' &= \frac{1}{3}\frac{\partial V}{\partial \beta}\,,\\
\gamma'' + 4\alpha'\gamma' &= \frac{1}{3}\frac{\partial V}{\partial \gamma}\,.
\end{aligned}
\tag{3.27}
$$

These equations are satisfied by our $AdS_5 \times \tilde{S}^3$ solutions (3.17) which, in terms of the new variables, read:

$$
\begin{aligned}
e^{\phi_0} &= \frac{2}{3}\frac{A}{Q_f}\,, \qquad & e^{\beta_0} &= Q_f^{\frac{1}{3}}\left(\frac{3}{32}\frac{(3-A)(A+1)^3}{A^2}\right)^{\frac{1}{6}}\,,\\
e^{\gamma_0} &= \left(\frac{3}{3-A}\right)^{\frac{1}{6}}\,, \qquad & \alpha &= \alpha_0 + \tilde{k}_0 u\,,
\end{aligned}
\tag{3.28}
$$

where $\alpha_0 = -\frac{1}{3}\phi_0 + \beta_0$, $\tilde{k}_0$ is related to k_0 in (3.17) through $\tilde{k}_0 = e^{\phi_0/3-\beta_0}k_0$ and, as before, A depends on the ratio Q_c/Q_f through the cubic equation (3.18).

The masses of (a subset of) the fluctuations around these solutions, are the eigenvalues of the laplacian acting on the small perturbations $\delta\alpha$, $\delta\phi$, $\delta\beta$ and $\delta\gamma$:

$$
\begin{aligned}
\alpha(u) &= \alpha_0 + \tilde{k}_0 u + \delta\alpha(u)\,,\\
\phi(u) &= \phi_0 + \delta\phi(u)\,,\\
\beta(u) &= \beta_0 + \delta\beta(u)\,,\\
\gamma(u) &= \gamma_0 + \delta\gamma(u)\,.
\end{aligned}
\tag{3.29}
$$

In our specific case, the laplacian reads:

$$
\nabla^2 f(u) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu f(u)) = e^{-\frac{2}{3}\phi+2\beta}(f'' + 2\phi' f' + 4\alpha' f')\,, \tag{3.30}
$$

and by substituting into this expression the equations of motion (3.27), we find, to first order in the perturbations,

$$
\begin{aligned}
\nabla^2(\delta\alpha) &= 0\,,\\
\nabla^2(\delta\phi) &= 3e^{-\frac{2}{3}\phi_0+2\beta_0}\left(\left.\frac{\partial^2 V}{\partial\phi^2}\right|_0\delta\phi + \left.\frac{\partial^2 V}{\partial\beta\partial\phi}\right|_0\delta\beta + \left.\frac{\partial^2 V}{\partial\gamma\partial\phi}\right|_0\delta\gamma\right),\\
\nabla^2(\delta\beta) &= \frac{1}{3}e^{-\frac{2}{3}\phi_0+2\beta_0}\left(\left.\frac{\partial^2 V}{\partial\phi\partial\beta}\right|_0\delta\phi + \left.\frac{\partial^2 V}{\partial\beta^2}\right|_0\delta\beta + \left.\frac{\partial^2 V}{\partial\gamma\partial\beta}\right|_0\delta\gamma\right),\\
\nabla^2(\delta\gamma) &= \frac{1}{3}e^{-\frac{2}{3}\phi_0+2\beta_0}\left(\left.\frac{\partial^2 V}{\partial\phi\partial\gamma}\right|_0\delta\phi + \left.\frac{\partial^2 V}{\partial\beta\partial\gamma}\right|_0\delta\beta + \left.\frac{\partial^2 V}{\partial\gamma^2}\right|_0\delta\gamma\right).
\end{aligned}
\tag{3.31}
$$

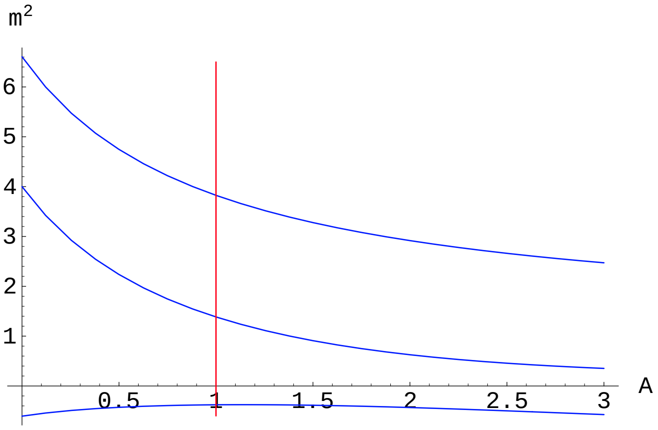

Figure 1: Squared masses of fluctuations around the AdS solutions. Notice that one of the eigenvalues is negative for all allowed values of A: $0 \leq A \leq 3$. The vertical line represents the special value $A = 1$.

Notice that at first order in the fluctuations, $\delta\alpha$ does not mix with the other fluctuations $\delta\phi$, $\delta\beta$ and $\delta\gamma$. The fluctuation $\delta\alpha$ is therefore trivially massless. We were not able to find a simple explicit form for the remaining eigenvalues of ∇^2, but we can at least evaluate them numerically. One thing can be noticed right away: the explicit dependence of the masses of the fluctuations on Q_f or Q_c cancels, and the spectrum only depends on their ratio Q_f/Q_c through the A-dependence[7]. For this reason, solutions corresponding to different values of Q_f/Q_c might have very different behavior. We plot in figure 1 the graph of the squared masses of the three non-trivial fluctuations we consider, as functions of A.

Notice that one of these three squared masses is always negative. Since the background we are considering is AdS, we have to ensure that this eigenvalue satisfies the Breitenlohner-Freedman (BF) bound in order for the solution to be stable [29]:

$$m^2 \geq -\frac{4}{R^2_{AdS}}\,, \tag{3.32}$$

where, in our case, $R^2_{AdS} = \frac{1}{k_0^2} = \frac{12(A+1)}{A+3}$. The BF stability condition is checked in figure 2, where the solid curve represents the squared mass of the lightest mode, and the dashed curve the extremal BF bound.

We notice that not all of our solutions satisfy (3.32): the BF bound is violated for all values $A > A_c$, where:

$$A_c = 2.625\ldots \tag{3.33}$$

which corresponds to:

$$\frac{Q_f}{Q_c} = 2.729\ldots \tag{3.34}$$

As we pointed out after (3.18), A is a monotonic function of Q_f/Q_c, therefore all backgrounds with Q_f/Q_c larger than the value (3.34) are unstable.

[7] This is different from what happens for the duals of $\mathcal{N} = 1$ superconformal QCD [10], where the masses of the fluctuations around the AdS solutions are completely independent of N_c and N_f.

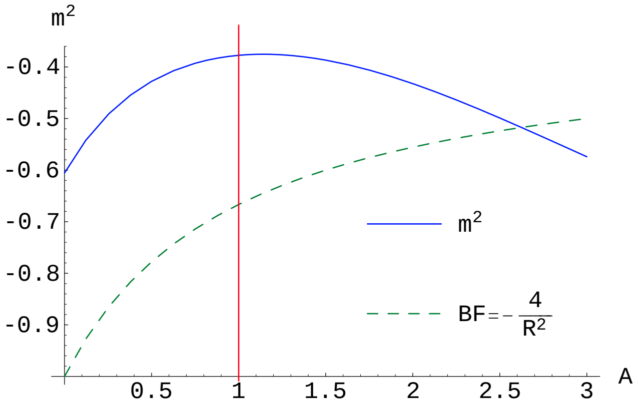

Figure 2: Stability of the solutions against the Breitenlohner-Freedman bound. The solid curve represents the squared mass of the lightest mode of fluctuations around the AdS solutions while the dashed line is the BF boundary. We notice that the bound is not satisfied for all allowed values of A. Again the vertical line shows the special case $A = 1$.

Fluctuation spectrum of the special solution $A = 1$

Since the special solution of section 3.3 corresponding to $Q_f/Q_c = \sqrt{2/3}$ (and $A = 1$) plays an important role, we focus here on this case. For the non-zero eigenvalues of (3.31) we find:

$$m_1^2 = -0.377\ldots \qquad m_2^2 = 1.385\ldots \qquad m_3^2 = 3.825\ldots \tag{3.35}$$

On the gauge theory side, a scalar fluctuation of mass m around a string background corresponds to a chiral operator with dimension[8] $\Delta = 2 \pm \sqrt{4 + m^2 R_{AdS}^2}$ [2, 3]. As we argued above, our solutions are subject to string corrections even at leading order, because the curvature of the background is always of the same order of magnitude as the string scale; therefore the spectrum (3.35) will receive corrections. Nonetheless, expecting that the structure of our AdS solutions will not be affected by string corrections, we anticipate that the higher order contributions we have neglected, will not correct drastically our results. Then, we can give a rough estimate of the conformal dimension of the operators dual to the fluctuations (3.29) around the $Q_f/Q_c = \sqrt{2/3}$ solution by using (3.35); we find ($R_{AdS}^2 = 6$):

$$\Delta_1 = 3.317\ldots \qquad \Delta_2 = 5.509\ldots \qquad \Delta_3 = 7.191\ldots \tag{3.36}$$

Regarding string corrections, it is interesting to notice that the laplacian eigenvector corresponding to the nearly vanishing eigenvalue m_1^2 is "dilaton dominated":

$$\delta\phi + (0.120\ldots)\delta\beta + (0.065\ldots)\delta\gamma \sim \delta\phi + (0.106\ldots)\delta\nu + (0.019\ldots)\delta f\,. \tag{3.37}$$

In ordinary critical AdS/CFT the dilaton is massless and it is dual to the marginal (i.e. $\Delta = 4$) operator Tr F^2 of the conformal gauge theory. In analogy to this, it is tempting to guess that string corrections will make the perturbation (3.37) massless, and therefore to identify this field with the field theory modulus associated to some marginal operator.

[8] For states with $m^2 R_{AdS}^2 > -3$ the only consistent choice of sign in the operator dimension formula is + [30].

4. Superpotential, BPS equations and a flow from AdS to KKL_4

The issue of conserved supersymmetries for the solutions we have found in section 3.2 is not at all trivial. Unfortunately, we lack a complete supersymmetric formulation of non-critical string theory on a background with RR fluxes, and in particular we do not know what the Killing spinor equations are. As we anticipated in section 3.3, in this section we will present the two main arguments which strongly suggest that the special solution $Q_f/Q_c = \sqrt{2/3}$ ($A = 1$) is supersymmetric: we will show that this is the only solution for which it is possible to write an explicit, simple superpotential, and we will find a numerical solution which confirms that the background with $A = 1$ is the back-reacted solution obtained by placing D3 and D5 branes on the $\mathbb{R}^{1,3}\times$ KKL_4 vacuum.

4.1 Superpotential

A necessary condition for supersymmetry is the existence of a superpotential [21, 22] whose related BPS equations are solved by the solutions we consider; studying this object will provide some important information about our solutions.

Let us start from the general case and suppose that the action of a system reads:

$$S = \int du\, e^{4\alpha} \left(3(\alpha')^2 - \frac{1}{2} G_{ab}(f) f'^a f'^b - V(f) \right). \tag{4.1}$$

The associated superpotential has to solve the following partial differential equation:

$$V = \frac{1}{8} G^{ab} \frac{\partial W}{\partial f^a} \frac{\partial W}{\partial f^b} - \frac{1}{3} W^2 . \tag{4.2}$$

When a solution to this equation does exist, the configurations satisfying the following set of first order BPS equations:

$$f'^a = \frac{1}{2} G^{ab} \frac{\partial W}{\partial f^b} , \qquad \alpha' = -\frac{1}{3} W(f) , \tag{4.3}$$

are solutions to the second order differential equations coming from (4.1) [21, 22]. Therefore finding a superpotential simplifies considerably the problem of building solutions.

Let us focus now on the particular case we are interested in, the solutions of the form (3.7), (3.8) to the effective action (3.6). By using the same redefinitions (3.23) as in section 3.4, we see that (3.25) is in the form (4.1), with (3.26):

$$V(\phi,\beta,\gamma) = -\frac{1}{2} e^{\frac{2}{3}\phi - 2\beta} + \frac{1}{8} e^{-4\beta - 8\gamma} - \frac{1}{2} e^{-4\beta - 2\gamma} + \frac{1}{4} Q_c^2 e^{\frac{2}{3}\phi - 8\beta} + \frac{1}{4} Q_f e^{\phi - 4\beta - 2\gamma} , \tag{4.4}$$

and (disregarding an overall factor):

$$G_{\phi\phi} = \frac{1}{3} , \qquad G_{\beta\beta} = 3 , \qquad G_{\gamma\gamma} = 3 , \tag{4.5}$$

and all other components vanishing. Equation (4.2) reads then:

$$\frac{3}{8} \left(\frac{\partial W}{\partial \phi} \right)^2 + \frac{1}{24} \left(\frac{\partial W}{\partial \beta} \right)^2 + \frac{1}{24} \left(\frac{\partial W}{\partial \gamma} \right)^2 - \frac{1}{3} W^2 = V(\phi,\beta,\gamma) . \tag{4.6}$$

We start solving this equation for the vacuum and therefore fix $Q_c = Q_f = 0$; it can be shown then that the superpotential:

$$W = -\frac{1}{2}e^{\frac{2}{3}\phi-2\gamma} - e^{-2\beta+2\gamma} - \frac{1}{2}e^{-2\beta-4\gamma} \tag{4.7}$$

is a solution to (4.6). We show in appendix B that the $\mathbb{R}^{1,3}\times$ KKL$_4$ vacuum (3.2), is the solution to the first order equations of motion we get from this superpotential.

Let us now turn on the RR flux and the flavor brane term. Remarkably, as anticipated in section 3.3, there is only one value of Q_f/Q_c for which the solution has a very simple form. This happens for:

$$\frac{Q_f}{Q_c} = \sqrt{\frac{2}{3}}\,, \tag{4.8}$$

in which case the superpotential is given by[9]:

$$W = -\frac{1}{2}e^{\frac{2}{3}\phi-2\gamma} - e^{-2\beta+2\gamma} - \frac{1}{2}e^{-2\beta-4\gamma} + Q_f e^{\frac{1}{3}\phi-4\beta}\,. \tag{4.9}$$

A very important feature of our solution appears here: we can find an explicit solution to (4.6) only for a single value of Q_f/Q_c. Indeed it is in general really hard, if possible, to find a superpotential that solves equation (4.6); for this reason, the result we find is quite remarkable, and very promising for the interpretation of our solutions.

As we argued before, the dual field theory to the special solution (4.8) should be an $\mathcal{N}=1$ SCFT where the requirements for conformal invariance and SU(2) global symmetry are satisfied only for a special value of N_f/N_c.

BPS first order equations for $Q_f/Q_c = \sqrt{2/3}$

With the superpotential (4.9) we found for the case $Q_f/Q_c = \sqrt{2/3}$, the first order equations of motion (4.3) read:

$$\begin{aligned}
\partial_u\alpha &= \frac{1}{6}e^{\frac{2}{3}\phi-2\gamma} + \frac{1}{3}e^{-2\beta+2\gamma} + \frac{1}{6}e^{-2\beta-4\gamma} - \frac{1}{3}Q_f\, e^{\frac{1}{3}\phi-4\beta}\,,\\
\partial_u\phi &= -\frac{1}{2}e^{\frac{2}{3}\phi-2\gamma} + \frac{1}{2}Q_f e^{\frac{1}{3}\phi-4\beta}\,,\\
\partial_u\beta &= \frac{1}{3}e^{-2\beta+2\gamma} + \frac{1}{6}e^{-2\beta-4\gamma} - \frac{2}{3}Q_f e^{\frac{1}{3}\phi-4\beta}\,,\\
\partial_u\gamma &= \frac{1}{6}e^{\frac{2}{3}\phi-2\gamma} - \frac{1}{3}e^{-2\beta+2\gamma} + \frac{1}{3}e^{-2\beta-4\gamma}\,.
\end{aligned} \tag{4.10}$$

Unfortunately, we were not able to find the general solution to these equations. Nonetheless, it is straightforward to show that the $AdS_5\times\tilde{S}^3$ solution with $Q_f/Q_c = \sqrt{2/3}$ (3.20) satisfies (4.10). The constant values of the fields read:

$$e^{\phi_0} = \frac{2}{3Q_f}\,, \qquad e^{2\beta_0} = Q_f^{2/3}\left(\frac{3}{2}\right)^{1/3}\,, \qquad e^{2\gamma_0} = \left(\frac{3}{2}\right)^{1/3}\,. \tag{4.11}$$

[9]Since equation (4.6) depends on W quadratically, the overall sign of the superpotential is not determined. We fix it by requiring that the AdS critical point is a local minimum of W rather than a maximum.

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4.2 A numerical flow from $AdS_5 \times \tilde{S}^3$ to the $\mathbb{R}^{1,3}\times$ KKL$_4$ vacuum

In this section we study the solution of the first order BPS equations (4.10) obtained by perturbing the special solution (4.11). As we will see, this will give rise to a domain wall interpolating between two different vacua of the eight-dimensional non-critical string theory: our solution $Q_f/Q_c = \sqrt{2/3}$ in the small radius region (IR) and the $\mathbb{R}^{1,3}\times$ KKL$_4$ vacuum in the large radius region (UV).

Before presenting the calculation, we will discuss what the interpretation of the flow we find should be. The domain wall represents the back-reaction of the stack of D3 and D5-branes we described in section 3.1, on the whole space-time and not just in the near-horizon limit. When we are in the IR region close to the D3-branes, the D5-branes are densely distributed on the transverse two-sphere and the background is our solution $AdS_5 \times \tilde{S}^3$, but as we go further away from this region the two-sphere grows and the D5-branes dilute. When we reach the large τ region (UV), the D3-branes are very far away and the density of the D5's is almost zero: the space-time feels a vanishing influence from the brane system and therefore it approaches the (empty) vacuum. Remarkably, in the UV region we find that the background asymptotes the $\mathbb{R}^{1,3}\times$ KKL$_4$ vacuum. This confirms that the solution with $Q_f/Q_c = \sqrt{2/3}$ is the near-horizon limit of the background generated by a stack of D3 and D5-branes placed on the KKL$_4$ vacuum.

Let us now construct the flow solution. For the following analysis, it is convenient to switch back to the radial coordinate τ. This amounts to the following redefinition of derivatives:

$$\dot{x} \equiv \partial_\tau x = e^{-\frac{1}{3}\phi+\beta}\partial_u x\,. \tag{4.12}$$

Since the field $\alpha(\tau)$ does not enter the equations (4.10) for the fields ϕ, β and γ, which are those we are interested in to match with the fluctuations of section 3.3, we will leave it aside for the moment and concentrate on:

$$\phi(\tau) = \phi_0 + \delta\phi(\tau)\,, \qquad \beta(\tau) = \beta_0 + \delta\beta(\tau)\,, \qquad \gamma(\tau) = \gamma_0 + \delta\gamma(\tau)\,. \tag{4.13}$$

We can expand the BPS equations (4.10) up to first order in the perturbations, and obtain:

$$\partial_\tau \begin{pmatrix} \delta\phi \\ \delta\beta \\ \delta\gamma \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -\frac{1}{3} & -4 & 2 \\ -\frac{4}{9} & \frac{8}{3} & \frac{2}{3} \\ \frac{2}{9} & \frac{2}{3} & -\frac{16}{3} \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta\beta \\ \delta\gamma \end{pmatrix}. \tag{4.14}$$

The eigenvalues of this matrix are given by:

$$\frac{1}{\sqrt{6}}(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{\sqrt{6}}(-0.683\ldots, -5.509\ldots, 3.191\ldots)\,. \tag{4.15}$$

The comparison of these eigenvalues to the conformal dimensions (3.36) will tell us whether on the dual field theory side these deformations amount to adding an operator or turning on a non-zero expectation value [2, 3]. To do so, we need to introduce the coordinate $z = \sqrt{6}e^{-\frac{\tau}{\sqrt{6}}}$, with respect to which the AdS metric reads $ds^2 = 6(dz^2 + dx_{1,3}^2)/z^2$. The first and second perturbations grow in the IR ($z \gg 1$) as $z^{-\lambda_1} = z^{4-\Delta_1}$ and $z^{-\lambda_2} = z^{\Delta_2}$ respectively, and therefore the first corresponds to adding a relevant (or marginal, see the comment at the end of section 3.4) operator to the action of the field theory, while the second one corresponds to giving an expectation value to an operator in the gauge theory. The third perturbation, on the contrary, grows in the UV ($z \sim 0$) as $z^{-\lambda_3} = z^{4-\Delta_3}$ and corresponds to adding an irrelevant operator to the action. Since we are interested in deforming the theory in the IR with an irrelevant operator, this last irrelevant, UV-growing deformation is the one we will turn on: we start in the IR with a configuration which asymptotes our AdS solution $(\phi_0, \beta_0, \gamma_0)$, and deform it with the eigenvector of (4.14) corresponding to λ_3 (4.15):

$$\phi = \phi_0 + \delta\phi\,, \qquad \beta = \beta_0 - (0.903\ldots)\delta\phi\,, \qquad \gamma = \gamma_0 - (0.045\ldots)\delta\phi\,, \tag{4.16}$$

which will drive the solution away from AdS in the UV. To study this flow, and eventually compare the UV asymptotic region with the $\mathbb{R}^{1,3}\times$ KKL$_4$ vacuum, it is convenient to analyze how the combinations of fields appearing in the metric (3.24) evolve, rather than the fields α, β and γ, that is to study the flow of:

$$\lambda = \frac{1}{3}\phi - \beta + \alpha\,, \qquad \nu = \frac{1}{3}\phi + \beta + \gamma\,, \qquad f = \frac{1}{3}\phi + \beta - 2\gamma\,, \tag{4.17}$$

and the dilaton ϕ. To compare the UV asymptotic behavior of the flow with the KKL$_4$ vacuum, we also introduce the same radial coordinate that enters the metric (3.2) for the vacuum:

$$d\zeta = 4\, e^{\frac{1}{3}\phi+\beta-2\gamma} d\tau\,. \tag{4.18}$$

The first order equations we need to solve read then:

$$\begin{aligned}
\partial_\zeta \phi &= -\frac{1}{8} + \frac{1}{8} Q_f e^{\phi-2\nu-2f}\,,\\
\partial_\zeta \nu &= \frac{1}{8} e^{-2\nu} - \frac{1}{8} Q_f e^{\phi-2\nu-2f}\,,\\
\partial_\zeta f &= -\frac{1}{8} + \frac{1}{4} e^{-2f} - \frac{1}{8} e^{-2\nu} - \frac{1}{8} Q_f e^{\phi-2\nu-2f}\,,\\
\partial_\zeta \lambda &= \frac{1}{8} Q_f e^{\phi-2\nu-2f}\,,
\end{aligned} \tag{4.19}$$

and the initial conditions we impose at $\zeta = 0$ (that is in the IR region) are (3.20):

$$\phi_0 = \log \frac{2}{3Q_f}\,, \qquad \nu_0 = 0\,, \qquad f_0 = \log \sqrt{\frac{2}{3}}\,, \qquad \lambda(0) = 0\,. \tag{4.20}$$

If we take $\delta\phi = 0$ there is obviously no flow, and we get our AdS solution on the whole space. If $\delta\phi$ is taken to be positive, the evolution of the fields blows up at some finite value of ζ. If we take $\delta\phi < 0$ the fields approach the KKL$_4$ vacuum value in the large ζ region. This requires the initial value of $\phi_{0,\mathrm{KKL}}$ to be tuned, which is a natural request if we think that in the vacuum the dilaton is linear and its value at $\zeta = 0$ is a free parameter, whereas in the AdS background the dilaton is fixed by the equations of motion. The numerical analysis is reported in figure 3.

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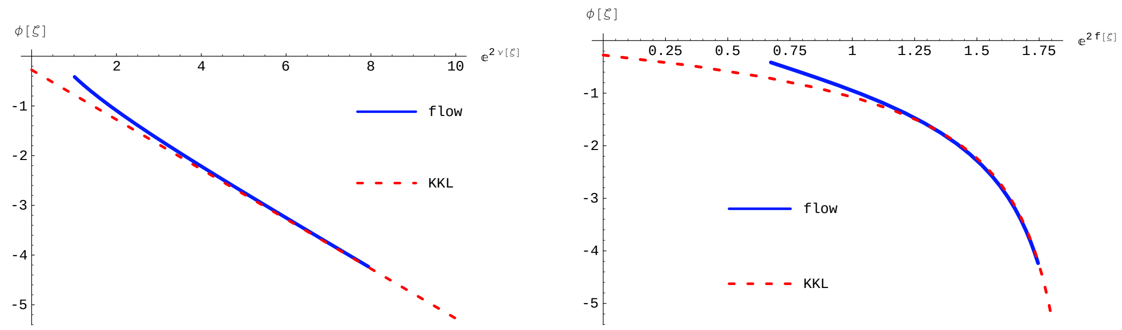

Figure 3: Relation between the dilaton $\phi(\zeta)$ and the fields $\nu(\zeta)$ (left) and $f(\zeta)$ (right) in the flow and in KKL$_4$. The flow starts from the $AdS_5 \times \tilde{S}^3$ solution. In both graphs the two curves overlap for decreasing ϕ, i.e. in the UV. The thicker line represents the flow solution, while the dashed line is the KKL$_4$ vacuum.

5. Comments on the $\mathcal{N} = 1$ dual field theory

It would be very important to identify the dual field theory to our special $AdS_5 \times \tilde{S}^3$ solution (3.20). Unfortunately, because of the intrinsic limitations of the gravity approximation to non-critical string theory, and a lack of knowledge of the low energy dynamics of branes on KKL$_4$, we are not able to give a final answer to this question.

As we argued in section 3.1, we believe that the $AdS_5 \times \tilde{S}^3$ solution (3.20) preserves eight supercharges (four background supercharges and four near-horizon conformal ones). This requires the dual field theory to be an $\mathcal{N} = 1$ superconformal gauge theory. The theory will contain flavors in the (anti) fundamental representation, and possibly other matter fields. Due to the symmetry of the dual background, it is conceivable that some SU(2) invariant superpotential for these fields will be present at the IR fixed point. This will also contribute to fix the N_f/N_c ratio.

Some hints toward an answer could come from the string engineerings of supersymmetric gauge theories and the description of NS5-brane configurations in terms of conformal field theories. It is well known [31] that an $\mathcal{N} = 2$ SU(N_c) gauge theory can be engineered by placing two parallel NS5-branes on flat space-time and attaching to them N_c D4-branes. Flavors can be introduced by adding D6-branes to the picture with additional D4-branes stretching from either NS-brane to any D6-brane. The general configuration is summarized in table 2.

A very interesting result, which is the base for the connection to non-critical string theories, states that a system of m separated NS5-branes is described by an exactly solvable CFT [32]:

$$\left(\frac{\mathrm{SL}(2,\mathbb{R})_{m+2}}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)_{m-2}}{\mathrm{U}(1)}\right)/\mathbb{Z}_m\,, \tag{5.1}$$

where $\frac{\mathrm{SL}(2,\mathbb{R})_{m+2}}{\mathrm{U}(1)}$ is the 2d cigar geometry, which has an asymptotically linear dilaton, while $\frac{\mathrm{SU}(2)_{m-2}}{\mathrm{U}(1)}$ stands for the $\mathcal{N} = 2$ minimal model with central charge $3(1 - \frac{2}{m})$. In particular,

	$x_{1,3}$				4	5	6	7	8	9
NS5	–	–	–	–	–	–	·	·	·	·
D4	–	–	–	–	·	·	–	·	·	·
D6	–	–	–	–	·	·	·	–	–	–

Table 2: Type-IIA configuration engineering $\mathcal{N} = 2$ QCD.

in the case $m = 2$ we are interested in, the central charge of the $\mathrm{SU}(2)/\mathrm{U}(1)$ conformal field theory is zero, and the target space directions which it describes play no role in the string theory dynamics. Attaching D-branes to the NS5-branes as in table 2 corresponds then to suitably placing D3 and D5-branes in a non-critical string theory background containing the cigar. We conjecture that considering eight-dimensional string theory on the vacuum:

$$\mathbb{R}^{1,3} \times \mathbb{R}^2 \times \frac{\mathrm{SL}(2,\mathbb{R})_4}{\mathrm{U}(1)}, \tag{5.2}$$

and placing (Minkowski-filling) D3 branes at the tip of the cigar gives rise to an $\mathcal{N} = 2$ gauge theory (for similar considerations see [33]). In particular $\mathcal{N} = 2$ superconformal QCD will be obtained by placing N_c D3-branes at the tip of the cigar, and $2N_c$ D5-branes completely wrapping the cigar. A first immediate check involves the Coulomb branch of the $\mathcal{N} = 2$ theory: the D3-branes cannot move from the tip of the cigar, but are free to move on the transversal $\mathbb{R}^2$, describing the classical Coulomb branch of an $\mathcal{N} = 2$ theory. Notice that the $\mathrm{SU}(2)_R$ symmetry is not visible as an isometry of the vacuum [34].

As we described in section 3.1, the vacuum where we place the branes is different from (5.2): our vacuum is the KKL_4 space of [20]. It has been shown in [27] that this space is related to parallel NS5-branes wrapping a vanishing $\mathbb{CP}^1$ inside the Eguchi-Hanson space, so the number of preserved supercharges is eight. The corresponding world-sheet CFT is given by [27]:

$$\left[\frac{\mathrm{LSM}_{4m}}{\mathbb{Z}_2} \times \frac{\mathrm{SU}(2)_{m-2}}{\mathrm{U}(1)}\right] / \mathbb{Z}_m^{\mathrm{diag}}, \tag{5.3}$$

where m is the number of NS5-branes and LSM_{4m} is the IR fixed point of a linear sigma model which describes KKL_4 with $K = 4m$ and therefore central charge $6(1 + \frac{1}{m})$.

Again, when we consider the case of $m = 2$ NS5-branes, two transverse directions are described by a zero central charge $\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$ conformal theory, and thus play no role in the string theory dynamics. Therefore, the $\mathcal{N} = 1$ gauge theory for D4 and D6-branes attached to these wrapped NS5-branes is equivalently described by D3 and D5-branes placed on $\mathbb{R}^{1,3} \times \mathrm{KKL}_4$.

Unfortunately, unlike the 2d cigar $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$, the conformal theory describing KKL_4 is not known to be integrable [27]. Nevertheless, this CFT has a mirror description in terms of the $\mathbb{Z}_m$ orbifold of a Landau-Ginzburg model with superpotential [27]:

$$W = e^{-mZ}\left(e^{-Y} + e^{Y}\right) + X^m, \tag{5.4}$$

where X, Y and Z are chiral superfields and the $\mathbb{Z}_m$ action is generated by $Z \to Z - \frac{2\pi i}{m}$, $X \to e^{\frac{2\pi i}{m}} X$, while Y is left invariant.

From this point of view, it was argued in [27] that the conformal field theory description of this NS5-brane configuration allows for a straightforward deformation of KKL_4 to be constructed. The LG orbifold has a finite set of exactly marginal operators $e^{-(m-l)Z}X^l$, for $l = 0, 1, \dots, m-2$. In particular for the case we are interested in, i.e. $m = 2$, there is a one-parameter deformation of (5.4):

$$W = e^{-2Z}\left(e^{-Y} + e^{Y}\right) + X^2 + e^{t/2}e^{-2Z} \,. \tag{5.5}$$

From the point of view of the NS5-branes, this deformation corresponds to taking the two branes away one from the other while at the same time inflating the $\mathbb{CP}^1$ they wrap [27]. The deformed superpotential (5.5) gives rise to a one-parameter deformation of the KKL_4 background. Remarkably all these spaces have the same large-radius behavior as KKL_4 (3.2). The two extrema of the family of deformations correspond to some special cases we already know: the $t \to -\infty$ limit obviously coincides with the unperturbed (5.3) configuration, while for $t \to +\infty$ we get the configuration corresponding to the superpotential [27]:

$$W = e^{-2Z'} + X^2 \,, \tag{5.6}$$

that is the mirror dual of two flat parallel NS5-branes.

Backed by these considerations, we argue that each theory described by placing branes on the one-parameter family of vacua dual to (5.5) could be connected to the $\mathcal{N} = 2$ theory which is obtained by placing branes on $\mathbb{R}^{1,3} \times \mathbb{R}^2 \times \frac{\mathrm{SL}(2,\mathbb{R})_4}{\mathrm{U}(1)}$.

D4-branes attached to the wrapped NS5-branes will have no possibility to move from their position when the NS5-branes wrap a vanishing cycle, while they will be free to move on a plane when the NS5-branes are flat. In all other intermediate cases, their motion will be delimited to the compact cycle wrapped by the NS5-branes. Since the position of the D4-branes on the NS5-branes is usually described by adjoint fields of the dual field theory, these will have a superpotential that reproduces the above behavior.

As we argued at the end of section 3.4, our $AdS_5 \times \tilde{S}^3$ back-reacted solution is expected to have a massless mode in its spectrum, corresponding to a marginal operator in the dual field theory. In view of the above description, we can interpret this massless mode as indicating on which one of the one-parameter family of deformed KKL_4 vacua we have placed D3 and D5-branes.

We report here also the following curious fact which, for the moment, lacking a deeper understanding of branes in non-critical string theory, we cannot consider as anything more than an intriguing accident. Our 8d background has the form:

$$ds^2 = \frac{R_{AdS}^2}{z^2}(dx_{1,3}^2 + dz^2) + R_{AdS}^2\, ds_3^2 \,, \tag{5.7}$$

where ds_3^2 is the metric of a supersymmetric three-cycle of $T^{1,1}$:

$$ds_3^2 = \frac{1}{9}(d\psi + \cos\theta d\varphi)^2 + \frac{1}{6}(d\theta^2 + \sin^2\theta d\varphi^2) \,. \tag{5.8}$$

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We have to remark, anyway, that α' corrections could well change the values of the eight-dimensional metric coefficients, and make the similarity disappear. If not, this could suggest a relation between the SCFT dual of our solution and the quiver Klebanov-Witten model dual to string theory on $AdS_5 \times T^{1,1}$ [35].

The problem of identifying the dual field theory would receive crucial hints from the study of branes on KKL_4 in a way similar to what was done for the cigar in [16, 17]. As we have already noticed, the conformal theory describing KKL_4 may not be integrable, and therefore this approach might prove very difficult. Nonetheless, considering the problem from the perspective of the dual Landau-Ginzburg description could be useful.

Let us conclude this section with some brief symmetry considerations. As a first observation, note that there is only one kind of D3-wrapped-on-$\tilde{S}^3$ we can construct in our case, i.e. only one kind of baryon vertex in the dual field theory. This requires the dual field theory to have baryons all with the same conformal dimension. Concerning the non-abelian flavor symmetry of the field theory, it is realized in the open string sector, while the baryonic $\mathrm{U}(1)_B$ should be dual, as usual, to the reduction of the four-form potential on $\tilde{S}^3$, the latter identification being in agreement with the baryon vertex just described. Finally, we can always switch on a constant axion on our background, playing the role of the θ_{YM} angle.

6. Fundamental flavors and problems for holography

The non-critical backgrounds found in this paper should *not* be taken as faithful *holographic gravity* duals of the corresponding $\mathcal{N}=0$ and $\mathcal{N}=1$ conformal field theories. In this sense these solutions suffer from the same limitations as those in [10]. These are due to the intrinsic limits of the two-derivative approximation to non-critical string theory, as for example the fact that the backgrounds have generically large curvature, or the lack of proofs that some supersymmetry is really preserved.

There is a simple known reason why we expect that conformal theories with fundamental (and antifundamental) flavors cannot have a weakly curved holographic dual. This is related to the structure of their gravitational central charges a and c. We examine this structure in the supersymmetric case where we can make use of exact results. In $\mathcal{N}=1$ terms, the gravitational central charges can be generically written as combinations of certain 't Hooft anomalies of the exact R-symmetry [36]:

$$a = \frac{3}{32}[3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R]\,, \quad c = \frac{1}{32}[9\,\mathrm{Tr}\,R^3 - 5\,\mathrm{Tr}\,R]\,, \tag{6.1}$$

so that:

$$a - c = \frac{1}{16}\,\mathrm{Tr}\,R\,. \tag{6.2}$$

For an $\mathcal{N}=1$ SCFT with gauge group $\mathrm{SU}(N_c)$, N_f chiral fields in the fundamental, N_f chiral fields in the anti-fundamental and N_A adjoint fields we have, at leading order in the large N_c limit (so that $N_c^2-1 \approx N_c^2$):

$$\mathrm{Tr}\,R = N_c^2 + N_c^2 \sum_{a=1}^{N_A}(R_a - 1) + N_c \sum_{f=1}^{N_f}(R_f - 1) + N_c \sum_{\tilde{f}=1}^{N_f}(R_{\tilde{f}} - 1)\,. \tag{6.3}$$

Here we have used the fact that the $\mathcal{N}=1$ vector multiplet has R-charge 1 (so the gauge field has $R=0$), and we have labeled with R_a the R-charge of the adjoint multiplets and with R_f, $R_{\tilde{f}}$ the ones of the fundamentals and anti-fundamentals.

Combining (6.3) with the relation imposed by the vanishing of the NSVZ beta function, it is easy to deduce the explicit expression:

$$a-c=\frac{N_c}{32}\left[\sum_{f=1}^{N_f}(R_f-1)+\sum_{\tilde{f}=1}^{N_f}(R_{\tilde{f}}-1)\right]. \tag{6.4}$$

This result is a non-perturbative, exact consequence of the exact formulae for the central charges and the β function, at large N_c, so that, in particular, we can compare it with dual string theory results. Notice that the adjoint fields never contribute to this expression.

From the formula above it follows that in a generic SCFT with fundamentals and anti-fundamentals, the non-zero difference $a-c$ may be leading in the large N_c limit. This is indeed what happens for the $\mathcal{N}=1,2$ conformal SQCD's [37]: for $\mathcal{N}=2$ conformal SQCD, at leading order, $a=\frac{7}{24}N_c^2$ and $c=\frac{N_c^2}{3}$, where N_f and N_c are of the same order of magnitude; for $\mathcal{N}=1$ conformal SQCD $a=\frac{3}{8}N_c^2[1-\frac{3}{2}\frac{N_c^2}{N_f^2}]$, $c=\frac{7}{16}N_c^2[1-\frac{9}{7}\frac{N_c^2}{N_f^2}]$.

Here we see a crucial difference from what happens in standard critical AdS/CFT examples, where the SCFT's have matter fields only in the bifundamental, adjoint and (anti)symmetric representations. In those cases, $\mathrm{Tr}\,R$ is subleading in N_c in the large N_c limit. This fact exactly matches with the holographic prediction $a=c$ from the dual $AdS_5\times X_5$ weakly curved backgrounds [7].

This holographic prediction automatically extends to every theory which has a classical weakly curved dual $AdS_5\times X_q$ background with constant dilaton ϕ:

$$a=c\approx V(X_q)e^{-2\phi}R_{AdS}^3\,, \tag{6.5}$$

where $V(X_q)$ is the volume of X_q and R_{AdS} is the AdS radius.

This result implies that superconformal theories with $N_f=O(N_c)$ fundamental fields cannot have a purely supergravity weakly curved holographic dual.[10] This is in agreement with what is found in this paper and in [10]: in both cases the supergravity backgrounds have large curvatures.

This argument can be extended to weakly coupled exactly conformal non-supersymmetric theories, for which the β-functions, a and c can be written in terms of the number of each kind of fields present in the theory. Although it is natural to expect that the same conclusion applies to general non-supersymmetric conformal fixed points, an analogous proof cannot be given in such cases.

In order to compute the value of the central charges of the theories above from the dual non-critical string models, one would therefore have to go beyond the supergravity limit. Just as a curiosity, formula (6.5) applied to the eight-dimensional background studied in this paper gives $a=c\approx N_c^2$. This result could eventually capture the correct scaling, but not the actual coefficients, of the central charges of a SCFT where N_f/N_c is finite.

[10]We are grateful to J. Sonnenschein, who was independently studying the same problems, for several discussions on the matter.

Apart from the central charges of the theory, other protected SCFT quantities that could be hopefully deduced from the dual non-critical string background are the baryon charges. Their stringy evaluation does not seem to suffer from the same intrinsic limitations encountered for the holographic calculation of the central charges. Just to focus on the supersymmetric example of this paper, it is expected that the baryonic $\mathrm{U}(1)_B$ symmetry of the dual $\mathcal{N}=1$ SCFT is holographically realized by the abelian field in AdS coming from the reduction of C_4 on the squashed three-sphere. Then, field theory baryons should be dual to D3 branes wrapped on $\tilde{S}^3$. The baryons are gauge invariant antisymmetrized products of N_c fundamental or anti-fundamental fields. Hence their conformal dimension is proportional to N_c. A wrapped D3 brane on $\tilde{S}^3$ has an effective mass given by:

$$m = T_3 V(\tilde{S}^3) e^{-\phi} = T_3 16\pi^2 Q_c\,, \tag{6.6}$$

where T_3 is the D3-brane tension. In the large mass approximation the conformal dimension of the dual operator is given by:

$$\Delta = R_{AdS} m = R_{AdS} T_3 16\pi^2 Q_c\,. \tag{6.7}$$

This result could reproduce the correct scaling of Δ with N_c, and could be used in principle to guess the value of R_{AdS} in the α' corrected solution provided the exact dual SCFT is identified.

7. Summary and outlook

In the last few years it has been realized that faithful string duals of some phenomenologically interesting theories, like pure Yang-Mills or QCD, involve string theories in less than ten dimensions.

In general, non-critical solutions have high curvature at the order of the string scale. This makes the task really difficult, since one should consider the full string theory in back-reacted, highly curved backgrounds with RR fields. This seems far from our present possibilities, but there have been some steps forward using two complementary approaches: in the first one [16, 17], world-sheet CFT techniques are used to study the physics of branes in the (non-backreacted) cigar, the relevant geometry to engineer the string dual of $\mathcal{N}=1$ SYM. In the second one [10, 12, 13], the two-derivative approximation is used in the belief that stringy corrections should not spoil the qualitative behavior of gravity solutions which are highly symmetric (like AdS solutions). This is of course an old argument [8, 9], but only recently it has been applied to theories with a large number of fundamental fields.

In this paper, we have further pursued this second approach. First, we have proposed a five-dimensional Anti de Sitter dual of conformal QCD by considering stacks of D3 color branes and D4-anti D4 flavor branes. The limits of the conformal window should correspond to the values of $\frac{N_f}{N_c}$ for which the dimension of the scalar meson operators hit the unitarity bound. We have also provided an interpretation of these meson operators from the world-volume dynamics of the flavor branes.

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Afterwards, we have explored the possibility of constructing a holographic dual by considering branes on a previously known four-dimensional space which we have dubbed KKL_4. Placing D3 branes at the tip of $\mathbb{R}^{1,3}\times$ KKL_4 along with D5 flavor branes appropriately embedded in the geometry, we have engineered the dual of an $\mathcal{N}=1$ conformal gauge theory with N_f chiral multiplets in the fundamental and anti-fundamental representations. Our results suggest that there is a supersymmetric, conformal ($AdS_5\times\tilde{S}^3$) solution for just one value of the ratio $\frac{N_f}{N_c}$. Finding the corresponding dual SCFT remains a challenge. It would be very interesting to study branes in KKL_4 à la [16, 17], since this would provide crucial information on the dual gauge theory.

As a general remark, we have provided further evidence that the two derivative approximation to non-critical strings can give some insight and qualitative agreement in the study of holographic pairs. However, it is not enough to get quantitative information about the dual field theories. For instance (see section 6), a difference of gravitational central charges $a-c$ must come in the string dual from higher derivative terms [7]. Since $a-c\neq 0$ in conformal theories with fundamental flavors, it is of obvious interest to develop tools that may allow us to study better such string duals.

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A. Explicit check of the 8d Einstein equations

The aim of this appendix is to write explicitly the set of relevant Einstein equations in the 8d case. Let us switch, for convenience, to Einstein frame $g_{\mu\nu}=e^{\frac{2}{3}\phi}g^E_{\mu\nu}$. The action (3.6) now reads:

$$S=\frac{1}{2\kappa^2_{(8)}}\int d^8x\sqrt{-g^E_{(8)}}\left(R-\frac{2}{3}(\partial_\mu\phi)^2+2e^{\frac{2}{3}\phi}-e^{-\frac{2}{3}\phi}F^2_{(5)}\right)+S^E_{\text{flavor}}\,, \tag{A.1}$$

where the action of the flavor branes is:

$$S^E_{flavor}=-\frac{T_5N_f}{4\pi}\int d^8x\sin\theta e^{\phi}\sqrt{-\hat{g}^E_{(6)}}\,. \tag{A.2}$$

We denote by $\hat{g}^E_{(6)}$ the determinant of the metric induced on the D5 flavor branes, which is simply:

$$d\hat{s}^2_E=e^{-\frac{2}{3}\phi}\left(e^{2\lambda}\,dx^2_{1,3}+d\tau^2+e^{2f}d\psi^2\right). \tag{A.3}$$

The Einstein equations read:

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \; & \frac{2}{3}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{3} g_{\mu\nu} (\partial\phi)^2 + g_{\mu\nu} e^{\frac{2}{3}\phi} + \\
& + \frac{1}{5!} e^{-\frac{2}{3}\phi} \left(5 F_{\mu\lambda_2\lambda_3\lambda_4\lambda_5} F_\nu^{\lambda_2\lambda_3\lambda_4\lambda_5} - \frac{1}{2} g_{\mu\nu} F_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} F^{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} \right) + \\
& + T^{\text{flavor}}_{\mu\nu} \,.
\end{aligned} \tag{A.4}
$$

The stress-energy tensor of the D5 branes can be computed as:

$$
T^{\mu\nu}_{flavor} = \frac{2\kappa^2_{(8)}}{\sqrt{-g^E_{(8)}}} \frac{\delta \mathcal{L}^E_{flavor}}{\delta (g^{(8)}_{\mu\nu})_E} = -\frac{\kappa^2_{(8)} T_5}{2\pi} N_f \sin\theta \, e^{\phi} \frac{1}{2} \delta^\mu_i \delta^\nu_j (\hat{g}^{(6)}_E)^{ij} \sqrt{\frac{\hat{g}^E_{(6)}}{g^E_{(8)}}} \,, \tag{A.5}
$$

where latin indices span the six dimensional world-volume. Inserting (3.11), one can readily write the components of the flavor stress-energy tensor:

$$
\begin{aligned}
T_{x_a x_b} &= -\frac{Q_f}{2} e^{2\lambda+\phi-2\nu} \eta_{ab} \,, \qquad & T_{\tau\tau} &= -\frac{Q_f}{2} e^{\phi-2\nu} \,, \\
T_{\psi\psi} &= -\frac{Q_f}{2} e^{\phi+2f-2\nu} \,, \qquad & T_{\varphi\psi} &= -\frac{Q_f}{2} e^{\phi+2f-2\nu} \cos\theta \,, \\
T_{\varphi\varphi} &= -\frac{Q_f}{2} e^{\phi+2f-2\nu} \cos^2\theta \,.
\end{aligned} \tag{A.6}
$$

One can also obtain the equation of motion of the dilaton from the action (A.1):

$$
\frac{1}{\sqrt{-g^E_{(8)}}} \partial_\mu \left(g^{\mu\nu}_E \sqrt{-g^E_{(8)}} \partial_\nu \phi \right) + e^{\frac{2}{3}\phi} + \frac{1}{2} e^{-\frac{2}{3}\phi} F^2_{(5)} - \frac{3}{4} Q_f e^{\frac{5\phi}{3} - 2\nu} = 0 \,. \tag{A.7}
$$

Finally, the equation for the five-form is:

$$
\partial_\mu \left(\sqrt{-g^E_{(8)}} e^{-\frac{2}{3}\phi} F^{\mu\lambda_2\lambda_3\lambda_4\lambda_5} \right) = 0 \,. \tag{A.8}
$$

It is straightforward to verify that equations (A.4), (A.7), (A.8) are satisfied by the general solution displayed in section 3.2 as well as by the $\mathbb{R}^{1,3} \times$ KKL$_4$ vacuum (3.2), once the respective metrics are transformed to the Einstein frame.

It is worth making a technical comment on the construction of the solution. Notice that the squashed three-sphere is not an Einstein space. In fact, the angular dependence in its Ricci tensor is not proportional to that in the metric tensor $g_{\mu\nu}$. However, for the given ansatz, all other terms in (A.4) except for the flavor one are indeed proportional to $g_{\mu\nu}$. Therefore, in order to satisfy the Einstein equations, $T^{flavor}_{\mu\nu}$ cannot have the same angular dependence as $g_{\mu\nu}$, as it is of course the case in (A.6). This is an effect of the smearing and would not happen for space-time filling D7 branes. Therefore, we conclude that an $AdS_5 \times \tilde{S}^3$ solution cannot be obtained by just using D7 instead of D5.

Finally, let us notice that when four-dimensional gauge theories are engineered in the ten dimensional IIA theory with branes suspended between branes, flavors are usually related to D6-branes. Since the non-critical backgrounds are related to the double scaling limits of such theories, it is much more natural to have in our setup D5-flavor branes rather than D7.

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B. Derivation of the $\mathbb{R}^2\times$ cigar and KKL$_4$ solutions from first order BPS equations

In this appendix we obtain the KKL$_4$ vacuum [20] as a solution of the first order BPS equations we derive from the vacuum superpotential (4.7), i.e. we take $Q_c = Q_f = 0$:

$$W = -\frac{1}{2}e^{\frac{2}{3}\phi-2\gamma} - e^{-2\beta+2\gamma} - \frac{1}{2}e^{-2\beta-4\gamma}\,. \tag{B.1}$$

We introduce the radial coordinate ζ such that $d\zeta = 4\,e^{\frac{1}{3}\phi+\beta-2\gamma}d\tau$, and use the same parametrization of the metric as in (3.7); the relation of this parametrization to the fields in (B.1) is shown in (4.17). The BPS first order equations read then:

$$\begin{aligned}
\partial_\zeta\phi &= -\frac{1}{8}\,,\\
\partial_\zeta\nu &= \frac{1}{8}e^{-2\nu}\,,\\
\partial_\zeta f &= -\frac{1}{8} + \frac{1}{4}e^{-2f} - \frac{1}{8}e^{-2\nu}\,,\\
\partial_\zeta\lambda &= 0\,.
\end{aligned} \tag{B.2}$$

The equations for the fields ϕ, ν and λ are trivial; remarkably, also the equation for f admits an analytic solution. We find:

$$\begin{aligned}
\phi(\zeta) &= \phi_0 - \frac{1}{8}\zeta\,,\\
e^{2\nu(\zeta)} &= e^{2\nu_0} + \frac{\zeta}{4}\,,\\
e^{2f(\zeta)} &= 8\frac{\frac{C}{8}e^{-\frac{\zeta}{4}} + \frac{\zeta}{4} - 1 + e^{2\nu_0}}{\zeta + 4e^{2\nu_0}}\,,\\
\lambda(\zeta) &= \lambda_0\,,
\end{aligned} \tag{B.3}$$

which is well defined only for:

$$\zeta \geq -4e^{2\nu_0}\,. \tag{B.4}$$

The integration constant ν_0 fixes the origin of the radial coordinate ζ, so we can fix it in such a way that $e^{2\nu_0} = 0$ and $\zeta \geq 0$. ϕ_0 is the value of the linear dilaton at $\zeta = 0$ and λ_0 is a scale factor which we can fix to 1 by a rescaling of the $\mathbb{R}^{1,3}$ coordinates. Finally the constant C has to be fixed to 8 in order for the solution to be regular at $\zeta = 0$.

The only thing that remains to be done is to evaluate the coefficient in front of the term $d\zeta^2$ in the metric. From the definition of ζ, $d\zeta = 4\,e^{\frac{1}{3}\phi+\beta-2\gamma}d\tau = 4\,e^{f}d\tau$, and the solution (B.3), it is easy to show that $d\tau^2 = \frac{\zeta}{128(e^{\frac{-\zeta}{4}}+\frac{\zeta}{4}-1)}d\zeta^2$. Collecting everything together, we can write the general solution to the vacuum BPS equations (B.2):

$$\begin{aligned}
ds^2 &= dx_{1,3}^2 + \frac{1}{4}\frac{\zeta}{32(e^{\frac{-\zeta}{4}} + \frac{\zeta}{4} - 1)}d\zeta^2 + \frac{\zeta}{4}(d\theta^2 + \sin^2\theta d\varphi^2) +\\
&\quad + \frac{1}{4}\frac{32(e^{\frac{-\zeta}{4}} + \frac{\zeta}{4} - 1)}{\zeta}(d\psi + \cos\theta d\varphi)^2\,,
\end{aligned}$$

$$\phi(\zeta) = \phi_0 - \frac{\zeta}{8} . \tag{B.5}$$

This is exactly (3.2) with $K = 8$, which is the right value to cancel the central charge for a background of the form $\mathbb{R}^{1,3}\times$ KKL$_4$ (see (3.5)).

Going back to (B.2), we can also consider a somehow singular solution of the first order equations. One can take $\nu(\zeta) = +\infty$ which corresponds to looking for a solution for which the S^2 base of the $\tilde{S}^3$ fibration has been blown up to a flat plane and therefore the squashed three-sphere has been substituted by the direct product of a plane and a circle parameterized by the angle ψ, with radius e^f. The equations for $\phi(\zeta)$ and $\lambda(\zeta)$ are not altered, so their solutions will be the same as in (B.3). The equation for $f(\zeta)$, instead, now reads:

$$\partial_\tau e^{-f} = \frac{1}{2} - e^{-2f} , \tag{B.6}$$

where we have gone back to the radial coordinate τ. The solution of this equation is:

$$e^{f(\tau)} = \sqrt{2}\, \tanh \frac{\tau}{\sqrt{2}} . \tag{B.7}$$

Rewriting the whole solution in terms of the variable τ we then have:

$$\begin{aligned} ds^2 &= dx_{1,3}^2 + dy_1^2 + dy_2^2 + d\tau^2 + 2\tanh^2 \frac{\tau}{\sqrt{2}} d\psi^2 , \\ e^{2\phi(\tau)} &= \frac{e^{2\phi_0}}{\cosh^2 \frac{\tau}{\sqrt{2}}} , \end{aligned} \tag{B.8}$$

which is the $\mathbb{R}^{1,3} \times \mathbb{R}^2 \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$ we claim to be the eight-dimensional vacuum where one needs to put branes to obtain the non-critical string dual of $\mathcal{N} = 2$ QCD. Notice that the BPS equations give us the cigar with the right CFT level to be a good eight-dimensional background of non-critical string theory.

C. Existence of a superpotential for generic N_f/N_c

We demonstrate in this appendix that a superpotential which solves equation (4.6) should exist for all values of N_f/N_c. This is an adaptation of an analogous argument of [12], where it is applied to backgrounds of the form $AdS_n \times S^k$, with S^k a round k-sphere.

Let us start from the effective action written as in (3.25); the superpotential W has then to satisfy equation (4.6):

$$\frac{3}{8}\left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{1}{24}\left(\frac{\partial W}{\partial \beta}\right)^2 + \frac{1}{24}\left(\frac{\partial W}{\partial \gamma}\right)^2 - \frac{1}{3}W^2 = V(\phi,\beta,\gamma) . \tag{C.1}$$

If a solution to this partial differential equation exists, that is if a superpotential W can be defined, then our general $AdS_5 \times \tilde{S}^3$ solution (3.28) must also solve the first order BPS equations of motion associated to W. In particular, then, we see from (4.3) that this implies:

$$\left.\frac{\partial W}{\partial \phi}\right|_0 = 0 , \qquad \left.\frac{\partial W}{\partial \beta}\right|_0 = 0 , \qquad \left.\frac{\partial W}{\partial \gamma}\right|_0 = 0 . \tag{C.2}$$

where $|_0$ stands for the value of the function at one of the $AdS_5 \times \tilde{S}^3$ solutions.

We notice that the choice of fields we have made is particularly convenient: the potential and the superpotential only depend on those fields which are set to a constant for solutions of the form $AdS_5 \times \tilde{S}^3$. By expanding both sides of (C.1) around the solution (3.28) we obtain the following equation for the derivatives of the superpotential at the AdS points:

$$\begin{aligned}
V|_0 + \ldots = & \frac{3}{8} \frac{\partial W}{\partial \phi}\Big|_0^2 + \frac{1}{24} \frac{\partial W}{\partial \beta}\Big|_0^2 + \frac{1}{24} \frac{\partial W}{\partial \gamma}\Big|_0^2 - \frac{1}{3} W|_0^2 + \\
& +2(\phi - \phi_0) \left(\frac{3}{8} \frac{\partial W}{\partial \phi}\Big|_0 \frac{\partial^2 W}{\partial \phi^2}\Big|_0 + \frac{1}{24} \frac{\partial W}{\partial \beta}\Big|_0 \frac{\partial^2 W}{\partial \phi \partial \beta}\Big|_0 + \right. \\
& \left. + \frac{1}{24} \frac{\partial W}{\partial \gamma}\Big|_0 \frac{\partial^2 W}{\partial \phi \partial \gamma}\Big|_0 - \frac{1}{3} W|_0 \frac{\partial W}{\partial \phi}\Big|_0 \right) + \\
& +2(\beta - \beta_0) \left(\frac{3}{8} \frac{\partial W}{\partial \phi}\Big|_0 \frac{\partial^2 W}{\partial \beta \partial \phi}\Big|_0 + \frac{1}{24} \frac{\partial W}{\partial \beta}\Big|_0 \frac{\partial^2 W}{\partial \beta^2}\Big|_0 + \right. \\
& \left. + \frac{1}{24} \frac{\partial W}{\partial \gamma}\Big|_0 \frac{\partial^2 W}{\partial \beta \partial \gamma}\Big|_0 - \frac{1}{3} W|_0 \frac{\partial W}{\partial \beta}\Big|_0 \right) + \\
& +2(\gamma - \gamma_0) \left(\frac{3}{8} \frac{\partial W}{\partial \phi}\Big|_0 \frac{\partial^2 W}{\partial \gamma \partial \phi}\Big|_0 + \frac{1}{24} \frac{\partial W}{\partial \beta}\Big|_0 \frac{\partial^2 W}{\partial \gamma \partial \beta}\Big|_0 + \right. \\
& \left. + \frac{1}{24} \frac{\partial W}{\partial \gamma}\Big|_0 \frac{\partial^2 W}{\partial \gamma^2}\Big|_0 - \frac{1}{3} W|_0 \frac{\partial W}{\partial \gamma}\Big|_0 \right) + \cdots , \qquad \text{(C.3)}
\end{aligned}$$

where $\cdots$ stand for quadratic terms, which are irrelevant for the present analysis. By matching the terms on both sides of the above equation, it is easy to show that a solution is given by:

$$\frac{\partial W}{\partial \phi}\Big|_0 = \frac{\partial W}{\partial \beta}\Big|_0 = \frac{\partial W}{\partial \gamma}\Big|_0 = 0 , \qquad -\frac{1}{3} W|_0^2 = V|_0 . \qquad \text{(C.4)}$$

Notice that the vanishing of the first derivatives of the superpotential at the $AdS_5 \times \tilde{S}^3$ solution does not depend on the specific value of the parameter A determining the solution, that is this argument is valid for every value of the ratio N_f/N_c. As a final check, let us evaluate the AdS radius. The BPS equation that fixes the radius of AdS is:

$$\partial_u \alpha = \partial_u \left(-\frac{2}{3}\phi + \frac{2}{3}\nu + \lambda + \frac{1}{3} f \right) = -\frac{1}{3} W . \qquad \text{(C.5)}$$

Since all fields are constant in the AdS solution but $\lambda(u)$, the above equation gives us the value of $\partial_u \lambda$. We only need to calculate the constant value of W as given by the last equation in (C.4) and switch back to the radial coordinate τ to compare our result with (3.17). We find:

$$k_0 = \partial_\tau \lambda = e^{-\frac{2}{3}\phi + \frac{2}{3}\nu + \frac{1}{3} f} \partial_u \lambda = \frac{1}{2\sqrt{3}} \sqrt{\frac{A+3}{A+1}} , \qquad \text{(C.6)}$$

which agrees with the value in (3.17) we had obtained through the second order equations of motion.

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D. Semiclassical analysis of D5-branes in $\mathbb{R}^{1,3}\times$ KKL$_4$

Some D-branes embedded in KKL$_4$ space were studied in [38]. Unfortunately, they do not include the kind of D5-branes that we are using to introduce flavor.

In this appendix we make a semiclassical analysis, based on the Dirac-Born-Infeld action of these branes. This is similar to what was done for D2 branes on the cigar [39], which was shown to be in good qualitative agreement with the exact solution [40, 41].

Let us consider a D5-brane extended along $x_0, \ldots, x_3, \zeta, \psi$ inside the geometry (3.1), (3.2) at constant θ, φ. It is easy to prove that this embedding is consistent with the DBI action. We then have, considering the excitation of the world-volume gauge field on the directions inside KKL$_4$:

$$\mathcal{L} = -T_5 e^{-\phi}\sqrt{-\det(P[g] + 2\pi F)} = -T_5\, e^{\frac{\zeta}{8}}\sqrt{\frac{1}{16} + (2\pi F_{\zeta\psi})^2}\,. \tag{D.1}$$

The equation of motion for A_ψ is $\frac{\partial \mathcal{L}}{\partial F_{\zeta\psi}} = const$ which renders:

$$2\pi F_{\zeta\psi} = \frac{1}{4\sqrt{e^{\frac{\zeta-\zeta_0}{4}} - 1}}\,, \tag{D.2}$$

where ζ_0 is a constant of integration. There are two, qualitatively different cases, for $\zeta_0 < 0$ and $\zeta_0 \geq 0$. In the first case, the D5 is extended up to the tip of the space $\zeta = 0$. In the second one, analogously to the D2-branes on the cigar considered in [41], the D5 terminates at a finite distance ζ_0 from the tip. This can be seen by noticing the vanishing of $T_{\zeta\zeta} = F_{\zeta\psi}\frac{\partial \mathcal{L}}{\partial F_{\zeta\psi}} - \mathcal{L}$ at that point. A non-vanishing F-field implies that the D5 brane carries some effective D3-brane charge, the total amount of which is given by $\int F_{\zeta\psi} d\zeta d\psi$. One can think of this object as a D3-D5 bound state. Since F vanishes for $\zeta \to \infty$, this D3 charge is concentrated at the bottom of the D5. This is also similar to the cigar case [40].

Notice that, for any value $\zeta_0 \geq 0$, the effective D3 charge is always the same. Furthermore, it is easy to prove that the energy of such configuration $E = \int -\mathcal{L}$, although infinite, does not depend on ζ_0, which is therefore a marginal deformation in this case (the same is not true for $\zeta_0 < 0$).

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[5] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *JHEP* **08** (2000) 052 [hep-th/0007191].

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[6] J.M. Maldacena and C. Núñez, *Towards the large-N limit of pure $N = 1$ super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001].

[7] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087].

[8] A.M. Polyakov, *The wall of the cave*, *Int. J. Mod. Phys.* **A 14** (1999) 645 [hep-th/9809057].

[9] A.M. Polyakov, *String theory as a universal language*, *Phys. Atom. Nucl.* **64** (2001) 540 [hep-th/0006132].

[10] I.R. Klebanov and J.M. Maldacena, *Superconformal gauge theories and non-critical superstrings*, *Int. J. Mod. Phys.* **A 19** (2004) 5003 [hep-th/0409133].

[11] M. Alishahiha, A. Ghodsi and A.E. Mosaffa, *On isolated conformal fixed points and noncritical string theory*, *JHEP* **01** (2005) 017 [hep-th/0411087].

[12] S. Kuperstein and J. Sonnenschein, *Non-critical supergravity ($d > 1$) and holography*, *JHEP* **07** (2004) 049 [hep-th/0403254].

[13] S. Kuperstein and J. Sonnenschein, *Non-critical, near extremal AdS_6 background as a holographic laboratory of four dimensional YM theory*, *JHEP* **11** (2004) 026 [hep-th/0411009].

[14] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *JHEP* **06** (2002) 043 [hep-th/0205236].

[15] M. Grana and J. Polchinski, *Gauge/gravity duals with holomorphic dilaton*, *Phys. Rev.* **D 65** (2002) 126005 [hep-th/0106014];
M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, $N = 2$ *gauge theories on systems of fractional D3/D7 branes*, *Nucl. Phys.* **B 621** (2002) 157 [hep-th/0107057];
J. Erdmenger and I. Kirsch, *Mesons in gauge/gravity dual with large number of fundamental fields*, *JHEP* **12** (2004) 025 [hep-th/0408113];
B.A. Burrington, J.T. Liu, L.A. Pando Zayas and D. Vaman, *Holographic duals of flavored $N = 1$ super Yang-Mills: beyond the probe approximation*, *JHEP* **02** (2005) 022 [hep-th/0406207].

[16] A. Fotopoulos, V. Niarchos and N. Prezas, *D-branes and SQCD in non-critical superstring theory*, hep-th/0504010.

[17] S.K. Ashok, S. Murthy and J. Troost, *D-branes in non-critical superstrings and minimal super Yang-Mills in various dimensions*, hep-th/0504079.

[18] H.J. Schnitzer, *Gauged vector models and higher-spin representations in AdS_5*, *Nucl. Phys.* **B 695** (2004) 283 [hep-th/0310210].

[19] D.J. Gross and F. Wilczek, *Asymptotically free gauge theories. I*, *Phys. Rev.* **D 8** (1973) 3633;
T. Banks and A. Zaks, *On the phase structure of vector - like gauge theories with massless fermions*, *Nucl. Phys.* **B 196** (1982) 189.

[20] E. Kiritsis, C. Kounnas and D. Lüst, *A large class of new gravitational and axionic backgrounds for four-dimensional superstrings*, *Int. J. Mod. Phys.* **A 9** (1994) 1361 [hep-th/9308124].

[21] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017].

[22] K. Skenderis and P.K. Townsend, *Gravitational stability and renormalization-group flow*, *Phys. Lett.* **B 468** (1999) 46 [hep-th/9909070];
O. DeWolfe, D.Z. Freedman, S.S. Gubser and A. Karch, *Modeling the fifth dimension with scalars and gravity*, *Phys. Rev.* **D 62** (2000) 046008 [hep-th/9909134];
J. de Boer, E.P. Verlinde and H.L. Verlinde, *On the holographic renormalization group*, *JHEP* **08** (2000) 003 [hep-th/9912012].

[23] G. Ferretti, J. Kalkkinen and D. Martelli, *Non-critical type 0 string theories and their field theory duals*, *Nucl. Phys.* **B 555** (1999) 135 [hep-th/9904013].

[24] A. Sen, *Dirac- Born-Infeld action on the tachyon kink and vortex*, *Phys. Rev.* **D 68** (2003) 066008 [hep-th/0303057].

[25] M.J. Strassler, *On confinement and duality*, prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2-10 Apr 2001.

[26] K. Zoubos, *A conformally invariant holographic two-point function on the Berger sphere*, *JHEP* **01** (2005) 031 [hep-th/0403292].

[27] K. Hori and A. Kapustin, *Worldsheet descriptions of wrapped NS five-branes*, *JHEP* **11** (2002) 038 [hep-th/0203147].

[28] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric* $\mathrm{SU}(N)\times\mathrm{SU}(N+M)$ *gauge theories*, *Nucl. Phys.* **B 578** (2000) 123 [hep-th/0002159].

[29] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys. (NY)* **144** (1982) 249.

[30] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104].

[31] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166];
A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, *Rev. Mod. Phys.* **71** (1999) 983 [hep-th/9802067].

[32] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164].

[33] D. Israel, A. Pakman and J. Troost, *D-branes in little string theory*, *Nucl. Phys.* **B 722** (2005) 3 [hep-th/0502073].

[34] S. Murthy, *Notes on non-critical superstrings in various dimensions*, *JHEP* **11** (2003) 056 [hep-th/0305197].

[35] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[36] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys.* **B 526** (1998) 543 [hep-th/9708042].

[37] D. Anselmi, *Quantum conformal algebras and closed conformal field theory*, *Nucl. Phys.* **B 554** (1999) 415 [hep-th/9811149];
D. Anselmi and A. Kehagias, *Subleading corrections and central charges in the AdS/CFT correspondence*, *Phys. Lett.* **B 455** (1999) 155 [hep-th/9812092].

[38] N. Nakayama and K. Sugiyama, *$N = 2$ supersymmetric sigma models and D-branes*, *Phys. Rev.* **D 70** (2004) 066003 [hep-th/0404227].

[39] A. Fotopoulos, *Semiclassical description of D-branes in* SL(2)/U(1) *gauged WZW model*, *Class. Quant. Grav.* **20** (2003) S465–S472 [hep-th/0304015].

[40] S. Ribault and V. Schomerus, *Branes in the 2-D black hole*, *JHEP* **02** (2004) 019 [hep-th/0310024].

[41] A. Fotopoulos, V. Niarchos and N. Prezas, *D-branes and extended characters in* $\mathrm{SL}(2,\mathbb{R})$*/u(1)*, *Nucl. Phys.* **B 710** (2005) 309 [hep-th/0406017].